Linear Systems, 2019 - Lecture 2

- Transition Matrix Properties
- Time-varying change of coordinates
- Periodic Systems
- Floquet Decomposition
- Time-varying Transfer Functions

Rugh, Chapter 5 [and Chapter 21]

Main news:

- Properties of LTV systems
- LTP systems

Continuous Time-varying (LTV) Systems

For bounded $A(t)$, the equation

$$\dot{x}(t) = A(t)x(t), \quad x(t_0) = x_0$$

has a unique solution of the form

$$x(t) = \Phi(t, t_0)x_0$$

The *transition matrix* $\Phi(t, t_0)$ can be written as the infinite sum

$$\begin{aligned}
\Phi(t, t_0) &= I + \int_{t_0}^{t} A(\sigma_1)d\sigma_1 \\
&\quad + \int_{t_0}^{t} A(\sigma_1) \int_{t_0}^{\sigma_1} A(\sigma_2)d\sigma_2 d\sigma_1 \\
&\quad + \int_{t_0}^{t} A(\sigma_1) \int_{t_0}^{\sigma_1} A(\sigma_2) \int_{t_0}^{\sigma_2} A(\sigma_3)d\sigma_3 d\sigma_2 d\sigma_1 \\
&\quad \ldots
\end{aligned}$$

Transition Matrix $\Phi(t, t_0)$

The unique solution of the equation

$$\begin{aligned} \frac{d}{dt}X(t) &= A(t)X(t) \\ X(t_0) &= I \end{aligned}$$

is $X(t) = \Phi(t, t_0)$.

Proof. Let $x(t) = X(t)x_0$. Then

$$\dot{x}(t) = \frac{d}{dt}X(t)x_0 = A(t)X(t)x_0 = A(t)x(t)$$

so

$$x(t) = \Phi(t, t_0)x_0$$

Hence $\Phi(t, t_0)x_0 = X(t)x_0$ for every x_0, so $\Phi(t, t_0) = X(t)$

Nice Example: Scalar Time-variation

Consider

$$\dot{x} = Aa(t)x(t)$$

The transition matrix is

$$\begin{aligned}
\Phi(t, t_0) &= I + A\int_{t_0}^{t} a(\sigma_1)d\sigma_1 + A^2\int_{t_0}^{t} a(\sigma_1)\int_{t_0}^{\sigma_1} a(\sigma_2)d\sigma_2 d\sigma_1 + \cdots \\
&= \sum_{k=0}^{\infty} \frac{1}{k!}A^k\left[\int_{t_0}^{t} a(\sigma)d\sigma\right]^k \\
&= \exp\left(A\int_{t_0}^{t} a(\sigma)d\sigma\right)
\end{aligned}$$

Second equality is nontrivial.

(Recall Two Tank Example with time-varying flow $q(t)$)

More general case: Commutating $A(t)$

If

$$A(t)\int_{t_0}^{t} A(\sigma)d\sigma = \int_{t_0}^{t} A(\sigma)d\sigma A(t)$$

then

$$\Phi(t, t_0) = \exp\left\{\int_{t_0}^{t} A(\sigma)d\sigma\right\}$$

Special case: $A(t)A(\tau) = A(\tau)A(t)$ for all t, τ

Example

If $A(t) = a_1(t)A_1 + a_2(t)A_2$ where A_1 and A_2 commute then

$$\begin{aligned}
\Phi(t, t_0) &= \exp\left\{\int_{t_0}^{t} a_1(t)A_1 + a_2(t)A_2 dt\right\} \\
&= \exp\left\{\int_{t_0}^{t} a_1(t)dtA_1\right\} \exp\left\{\int_{t_0}^{t} a_2(t)dtA_2\right\}
\end{aligned}$$

Example

$$
\begin{aligned}
\dot{x}(t) &= \begin{bmatrix} 1 & \cos t \\ 0 & 0 \end{bmatrix} x(t) \\
x_2(t) &\equiv x_2(\tau) \\
\dot{x}_1(t) &= x_1(t) + \cos t \cdot x_2(\tau) \\
x_1(t) &= e^{t-\tau} x_1(\tau) + \int_\tau^t e^{t-\sigma} \cos\sigma d\sigma \cdot x_2(\tau) \\
&= e^{t-\tau} x_1(\tau) + \frac{1}{2}\Big(\sin t - \cos t - e^{t-\tau}(\sin\tau - \cos\tau)\Big) \cdot x_2(\tau) \\
\Phi(t,\tau) &= \begin{bmatrix} e^{t-\tau} & \frac{1}{2}\left(\sin t - \cos t - e^{t-\tau}(\sin\tau - \cos\tau)\right) \\ 0 & 1 \end{bmatrix}
\end{aligned}
$$

Sanity check: $\Phi(t,t) = I$ and $\frac{d}{dt}\Phi(t,\tau)\Big|_{t=\tau} = \begin{bmatrix} 1 & \cos t \\ 0 & 0 \end{bmatrix}$

Input-driven Continuous System

The equation

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ x(t_0) &= x_0 \end{aligned}$$

has the unique solution

$$x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, \sigma)B(\sigma)u(\sigma)d\sigma$$

Proof: Differentiate!

Properties of $\Phi(t, \sigma)$

For any t, τ, σ, the transition matrix satisfies

$$\begin{aligned} \Phi(t,\tau) &= \Phi(t,\sigma)\Phi(\sigma,\tau) \text{ (semigroup property)} \\ \frac{d}{dt}\Phi(t,\sigma) &= A(t)\Phi(t,\sigma) \\ \frac{d}{d\sigma}\Phi(t,\sigma) &= -\Phi(t,\sigma)A(\sigma) \end{aligned}$$

Proof of first property: Let $R(t) = \Phi(t,\sigma)\Phi(\sigma,\tau)$. Then

$$\begin{aligned} \frac{d}{dt}R(t) &= A(t)R(t) \\ R(\sigma) &= \Phi(\sigma,\tau) \end{aligned}$$

so $R(t)$ must be identical to $\Phi(t,\tau)$

Properties of $\Phi(t, \sigma)$

Proof of third property:

$$\Phi(\sigma + h, \sigma) = I + hA(\sigma) + o(h) \qquad \text{(why?)}$$

Hence, using first property, we have

$$\Phi(t, \sigma) = \Phi(t, \sigma + h)(I + hA(\sigma) + o(h))$$

from which we get

$$\frac{1}{h}(\Phi(t, \sigma + h) - \Phi(t, \sigma)) = -\Phi(t, \sigma + h)A(\sigma) + o(1)$$

from which the result follows as $h \to 0$

$$\frac{d}{d\sigma}\Phi(t, \sigma) \;\; = \;\; -\Phi(t, \sigma)A(\sigma)$$

Inversion

The transition matrix $\Phi(t, t_0)$ is invertible for any t, t_0 and

$$\Phi(t, t_0)^{-1} \quad = \quad \Phi(t_0, t)$$

Proof. By the composition rule

$$\Phi(t, t_0)\Phi(t_0, t) = \Phi(t_0, t)\Phi(t, t_0) = \Phi(t_0, t_0) = I$$

Short summary: Properties of $\Phi(t, \sigma)$

For any t, τ, σ, the transition matrix satisfies

$$\begin{aligned}
\Phi(t,t) &= I \\
\Phi(t,\tau) &= \Phi(t,\sigma)\Phi(\sigma,\tau) \\
(\Phi(t,\sigma))^{-1} &= \Phi(\sigma,t) \\
\frac{d}{dt}\Phi(t,\sigma) &= A(t)\Phi(t,\sigma) \\
\frac{d}{d\sigma}\Phi(t,\sigma) &= -\Phi(t,\sigma)A(\sigma)
\end{aligned}$$

Change of Variables

Variable change $x(t) = P(t)z(t)$ (with $P(t)$ invertible) gives

$$\begin{aligned}
\dot{x}(t) &= A(t)x(t), \quad x(t_0) = x_0 \qquad \Longleftrightarrow \\
\dot{z}(t) &= \left[P(t)^{-1}A(t)P(t) - P(t)^{-1}\dot{P}(t)\right]z(t), \quad z(t_0) = P(t)^{-1}x_0
\end{aligned}$$

For the fundamental matrix this means that

$$\Phi_{P^{-1}AP-P^{-1}\dot{P}}(t,t_0) = P^{-1}(t)\Phi_A(t,t_0)P(t_0)$$

Proof:

$$\begin{aligned}
APz = Ax &= \dot{x} = \dot{P}z + P\dot{z} \\
z(t) &= \Phi_{P^{-1}AP-P^{-1}\dot{P}}(t,t_0)z(t_0) \\
z(t) &= P^{-1}(t)x(t) = P^{-1}(t)\Phi_A(t,\tau)P(t_0)z(t_0)
\end{aligned}$$

Adjoint system

From $\frac{d}{d\sigma}\Phi_A(t,\sigma) = -\Phi_A(t,\sigma)A(\sigma)$ follows that $\Phi_A^T(t,\sigma)$ solves

$$\frac{d}{d\sigma}Z(\sigma) = -A^T(\sigma)Z(\sigma), \quad Z(t) = I$$

This fact can be written as

$$\Phi_{-A^T}(t,t_0) = \Phi_A^T(t_0,t)$$

Discrete time - Definition of $\Phi(k, k_0)$

Define $X(k)$ recursively as

$$\begin{aligned} X(k+1) &= A(k)X(k), \quad k \geq k_0 \\ X(k_0) &= I \end{aligned}$$

Then $\Phi(k, k_0) = X(k)$.

Remark: What about $\Phi(k, k_0)$ when $k < k_0$?

The example $x(k+1) = 0 \cdot x(k)$ shows that $x(k)$ might not be uniquely determined by $x(k_0)$ for $k < k_0$!

Difference between discrete and continuous time

Properties of $\Phi(k, k_0)$

$$\begin{aligned}
\Phi(k+1, j) &= A(k)\Phi(k, j), \quad k \geq j \\
\Phi(k, j-1) &= \Phi(k, j)A(j-1), \quad k \geq j \\
\Phi(k, i) &= \Phi(k, j)\Phi(j, i), \quad k \geq j \geq i
\end{aligned}$$

If the $n \times n$ matrix $A(k)$ is invertible for each k, then $\Phi(k, j)$ is invertible for each $k \geq j$ and $\Phi(j, k)$ can be defined as

$$\Phi(j, k) = \Phi(k, j)^{-1}$$

Change of Variables

Variable change $x(k) = P(k)z(k)$ (with $P(k)$ invertible) we get

$$\begin{aligned} x(k+1) &= A(k)x(k), \quad x(k_0) = x_0 \qquad \iff \\ z(k+1) &= \left[P(k+1)^{-1}A(k)P(k)\right] z(k) \end{aligned}$$

Hence we have

$$\Phi_z(k,j) = P(k)^{-1}\Phi_x(k,j)P(j)$$

Theorem by Abel-Jacobi-Liouville

Let $A(t)$ be continuous. Then

$$\det \Phi(t, t_0) \quad = \quad \exp\left(\int_{t_0}^{t} \mathrm{tr}[A(\sigma)] d\sigma\right)$$

Interpretation: Volume contraction

Proof: Let c_{ij} be the cofactor of entry ϕ_{ij}

$$\begin{aligned}
\frac{d}{dt}\det\Phi(t, t_0) &= \sum_{i,j}\left(\frac{\partial}{\partial\phi_{ij}}\det\Phi(t, t_0)\right)\dot{\phi}_{ij}(t, t_0) \\
&= \sum_{i,j} c_{ij}(t, t_0)\dot{\phi}_{ij}(t, t_0) \\
&= \mathrm{tr}\left(C(t, t_0)^T\dot{\Phi}(t, t_0)\right) \\
&= \mathrm{tr}\left(\Phi(t, t_0)C(t, t_0)^T A(t)\right) \\
&= \mathrm{tr}\left((\det\Phi(t, t_0)I)A(t)\right) \\
&= \mathrm{tr}A(t) \cdot \det\Phi(t, t_0)
\end{aligned}$$

Example - From Exam 2009

Is it possible to asymptotically stabilize the oscillative system

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{aligned}$$

by time-varying output feedback

$$u(t) = -l(t)y(t)?$$

Solution

The closed loop system becomes $\dot{x} = A_c x$ with

$$A_c = \begin{bmatrix} 0 & 1 \\ -1 - l(t) & 0 \end{bmatrix}$$

By the Abel-Liouville theorem we have

$$\det \Phi(t, 0) = \exp(t\, \mathrm{tr} A_c) \equiv 1$$

The system can hence not be asymptotically stable, since an asymptotically stable system must have $\Phi(t, 0) \to 0$ as $t \to \infty$ [Why?].

Example - LTV systems and Eigenvalues

Consider the time-varying system

$$\dot{x} = e^{-At} B e^{At} x \tag{1}$$

Note that $e^{-At} B e^{At}$ has the same eigenvalues as B

The coordinate change $z(t) = e^{At} x(t)$ transforms the system to

$$\dot{z} = (A + B) z \tag{2}$$

Example - LTV Systems and Eigenvalues

Assume

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Then with $z(t) = e^{At}x(t)$ one has $\|z\| = \|x\|$, so asymptotic stability of (1) and (2) are equivalent (rotating coordinate transformation).

Proof: More generally if $A(t)$ is skew-symmetric for all t, i.e. $A^T + A = 0$ then $Q(t) = \Phi_A(t, 0)$ is orthogonal, i.e. satisfies $Q^T Q = I$, since

$$\frac{d}{dt} Q^T Q = Q^T (A^T + A) Q = 0, \qquad Q(0) = I$$

Therefore $z^T z = x^T Q^T Q x = x^T x$.

Example - LTV Systems and Eigenvalues

With the stable matrix

$$B = \begin{bmatrix} -1 & M \\ 0 & -1 \end{bmatrix}$$

it is easy to see that $A + B$ is unstable for $M > 2$.

Hence system (1) above is an **unstable time-varying system with stable eigenvalues** (equal to -1 for all t).

Example -LTV Systems and Eigenvalues

With the unstable matrix

$$B = \begin{bmatrix} -1 & 0 \\ 0 & 1/2 \end{bmatrix}$$

it is easy to see that $A + B$ is stable.

Hence system (1) above is a stable LTV system having one unstable eigenvalue for all t.

Exercise: Can you find a 2×2 asymptotically stable LTV system with both eigenvalues in the RHPL for all t?

Linear Time Periodic (LTP) Systems

A linear system

$$\dot{x}(t) = A(t)x(t)$$

with

$$A(t+T) = A(t)$$

is said to be T-periodic.

The smallest such T is called the period of the system.

A state space system is called T-periodic if all matrices (A, B, C, D) are T-periodic.

The following is the main result for periodic systems

Floquet Decomposition

"Long-term trend + periodic fluctuations"

Let $A(t)$ be bounded and T-periodic. Then for

$$\dot{x}(t) = A(t)x(t), \quad x(t_0) = x_0$$

the transition matrix can be written

$$\Phi(t, \tau) = P(t)e^{R(t-\tau)}P(\tau)^{-1}$$

where $R \in \mathbf{C}^{n \times n}$ is constant and $P(t) \in \mathbf{C}^{n \times n}$ is differentiable, invertible, and T-periodic.

The variable transformation $x(t) = P(t)z(t)$ gives $\dot{z} = Rz$

Proof

Since $\Phi(T,0)$ is nonsingular, there exists a solution $F \in C^{n\times n}$ (in fact infinitly many) to $e^F = \Phi(T,0)$. Choosing any such F, define $R = \frac{1}{T}F$, we then have

$$e^{RT} = \Phi(T,0)$$

Define then $P(t)$ by

$$P(t) = \Phi(t,0)e^{-Rt}$$

We get

$$\begin{aligned}
\Phi(t,\tau) &= \Phi(t,0)\Phi(\tau,0)^{-1} = P(t)e^{R(t-\tau)}P(\tau)^{-1} \\
P(t+T) &= \Phi(t+T,0)e^{-R(t+T)} \\
&= \Phi(t+T,T)\Phi(T,0)e^{-RT}e^{-Rt} \\
&= \Phi(t+T,T)e^{-Rt} \\
&= \Phi(t,0)e^{-Rt} \\
&= P(t)
\end{aligned}$$

Discrete Time Floquet Decomposition

Let $A(k)$ be K-periodic. Then for

$$x(k+1) = A(k)x(k), \quad x(k_0) = x_0$$

the transition matrix can be written

$$\Phi(k,j) = P(k)R^{(k-j)}P(j)^{-1}$$

where $R \in \mathbf{C}^{n\times n}$ and $P(k)$ is K-periodic.

With $x(k) = P(k)z(k)$, this gives

$$z(k+1) \quad = \quad Rz(k)$$

2-periodic Example

$$\begin{aligned} A(k) &= \begin{bmatrix} (-1)^k & 0 \\ 0 & 1 \end{bmatrix} \\ R^2 = \Phi(2,0) &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \\ R &= \begin{bmatrix} i & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Note that R is not real-valued !

Extra: Real Floquet Factors for LTP Systems

(not in Rugh)

It is always possible to obtain a **real** Floquet factorisation for a real T-periodic system, by treating the system as having 2T-periodic coefficients:

From the fact that $\Phi(2T,0) = \Phi(T,0)^2$ it can be proved (use Jordan-form) that there is a real matrix G such that

$$e^{2TG} = \Phi(2T,0).$$

Then $P(t) := \Phi(t,0)e^{-tG}$ is real and can as before be seen to be $2T$-periodic (but not necessarily T-periodic).

See Montagnier, P, et.al Real Floquet Factors of Linear Time-Periodic Systems (Google it)

LTI with sinusodal input - Resonances

Consider the equation

$$\dot{x} \quad = \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ \sin t \end{bmatrix} \quad x(0) = x_0$$

Laplace transform:

$$\begin{aligned} \mathbf{x}_2(s) &= C(sI - A)^{-1}(Bu(s) + x_0) \\ &= \frac{s}{(1+s^2)^2} + \frac{1}{1+s^2} \begin{bmatrix} 1 & s \end{bmatrix} x_0 \end{aligned}$$

$$x_2(t) \quad = \quad \frac{t}{2}\sin t + \begin{bmatrix} \sin t & \cos t \end{bmatrix} x_0$$

For what systems does periodic input give periodic solution?

Periodic Solutions for LTI Systems

For $A \in \mathbf{R}^{n \times n}$ and

$$\dot{x}(t) = Ax(t) + f(t)$$

one can prove that the following statements are equivalent:

(i) No eigenvalue of A has zero real part.

(ii) A unique T-periodic solution exists for every T-periodic f.

Periodic Solutions for LTP Systems

Theorem 5.15

Let $A(t)$ be continuous and T-periodic and

$$\dot{x}(t) = A(t)x(t) + f(t)$$

The following statements are then equivalent:

(i) No nontrivial T-periodic solution exists for $f \equiv 0$.

(ii) A unique T-periodic solution exists for every T-periodic f.

Time-varying Transfer Functions

(not in Rugh)

Transfer function analysis is quite involved for time-varying linear systems

$$y(t) = \int h(t,\tau)u(\tau)d\tau$$

For LTI systems, $h(t,\tau) = h(t-\tau,0)$ and

$$Y(j\omega) = H(j\omega)U(j\omega), \qquad H(j\omega) = \int h(r,0)e^{-j\omega r}dr$$

What can be said for general LTV systems?

And for LTP systems, where $h(t+T,\tau+T) = h(t,\tau)$?

Transfer Functions for LTV Systems

Define as usual

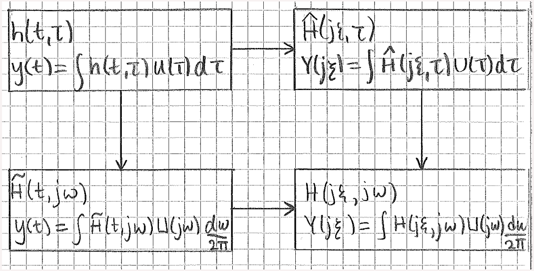
$$U(j\omega) = \int e^{-j\omega\tau} u(\tau) d\tau; \qquad u(t) = \int e^{j\omega\tau} U(j\omega) d\omega / 2\pi$$
$$Y(j\xi) = \int e^{-j\xi t} y(t) dt; \qquad y(t) = \int e^{j\xi t} Y(j\xi) d\xi / 2\pi$$

Define also

$$\begin{aligned} \widehat{H}(j\xi, \tau) &= \int e^{-j\xi t} h(t, \tau) dt \\ \widetilde{H}(t, j\omega) &= \int e^{j\omega\tau} h(t, \tau) d\tau \\ H(j\xi, j\omega) &= \iint e^{j\omega\tau - j\xi t} h(t, \tau) d\tau dt \end{aligned}$$

We then have the following relations (if convergence ok)

Transfer Functions for LTV Systems



For LTI systems we have

$$H(j\xi, j\omega) = 2\pi\delta(\xi - \omega)H(j\omega)$$

and

$$Y(j\omega) = H(j\omega)U(j\omega)$$

Transfer Functions for LTP Systems

If $h(t+T, \tau+T) = h(t, \tau)$ then (with $\omega_0 = 2\pi/T$)

$$Y(j\omega) = \sum_k H_k(j\omega - jk\omega_0)U(j\omega - jk\omega_0)$$

where

$$H_k(j\omega) = \mathsf{F}(h_k(t)) = \int e^{-j\omega t} h_k(t) dt$$

with

$$h_k(t) = \frac{1}{T}\int_0^T h(r, r-t) e^{-jk\omega_0 r} dr$$

The interpretation is that H_k describes the signal transfer from input frequency ω to output frequency $\omega + k\omega_0$.

Next Week

- Controllability and Observability
- Controller/Observer Forms
- Balanced Realizations

Bonus: Abel-Jacobi-Liouville alt. proof

Want to prove that

$$\frac{d}{dt}\det\Phi(t,t_0) \quad = \quad \mathrm{tr}A(t)\det\Phi(t,t_0)$$

Taylor-expansion gives

$$\begin{aligned}\Phi(t+dt,t_0) &= \Phi(t,t_0) + A(t)\Phi(t,t_0)dt + o(dt) \\ &= (I + A(t)dt)\Phi(t,t_0) + o(dt)\end{aligned}$$

Since the determinant is the product of the eigenvalues, and these satisfy $\lambda_i(I + Adt) = 1 + \lambda_i(A)dt$ we get

$$\det\Phi(t+dt,t_0) \quad = \quad \prod(1 + \lambda_i(A)dt) \cdot \det\Phi(t,t_0) + o(dt)$$
$$= (1 + \sum \lambda_i(A)dt + o(dt)) \cdot \det\Phi(t,t_0) + o(dt)$$

From which the result follows by letting $dt \to 0$.

Bonus: The Oscillating Inverted Pendulum

The unstable equilibrium becomes stable if the point of suspension oscillates fast in the vertical direction:

Pendulum length l

Oscillation amplitude $a << l$

Period of oscillation 2τ

Acceleration supposed *constant* equal to $\pm c$ (so $c = 8a/\tau^2$)

Assume

$$\ddot{x} = (\omega^2 \pm d^2)x$$

where the sign changes after time τ, where $\omega^2 = g/l$ and $d^2 = c/l$. If the oscillation of the suspension is fast enough, $c > g$, then $d^2 = 8a/(l\tau^2) > \omega^2$.

Bonus: The Oscillating Inverted Pendulum

If you have time over:

[a.] Show that $\Phi(2\tau, 0) = A_2 A_1$

$$A_1 = \begin{bmatrix} \cosh k\tau & 1/k \sinh k\tau \\ k \sinh k\tau & \cosh k\tau \end{bmatrix}, \quad A_2 = \begin{bmatrix} \cos\Omega\tau & 1/\Omega \sin\Omega\tau \\ -\Omega \sin\Omega\tau & \cos\Omega\tau \end{bmatrix}$$

where $k^2 = d^2 + \omega^2$ and $\Omega^2 = d^2 - \omega^2$.

[b.] Show that det$(\Phi(t, 0)) = 1$, $t = 2\tau, 4\tau, 6\tau, \ldots$ and that the criterion for stability is $|\text{tr}(\Phi(2\tau, 0))| < 2$

[c.] Suppose that $a << l$. Show that the system is stable for sufficiently large $c >> g$. The stability condition is approximately

$$c/g > (3/2)\, l/a$$