Lecture 8

- Differential Algebraic Equations
- Rosenbrock System Matrix
- Course Review

Suggested reading: T. Kailath *Linear Systems*, Chapter 8 (link available in the email).

Differential Algebraic Equation

Models of physical systems are often on the form

$$0 = F(\dot{x}, x, t)$$

If x and $\dot{x}$ enter linearly we get

$$E\dot{x} = Ax + f(t)$$

Linear **Differential Algebraic Equation (DAE)**

Any linear differential equation with higher order derivatives can be brought into this form by augmenting the state vector.

E might not be invertible

Example: Two Tank System

Flow: q, Volumes: V_1, V_2, Concentrations: $u(t)$, $x_1(t)$, $x_2(t)$

Dynamics:

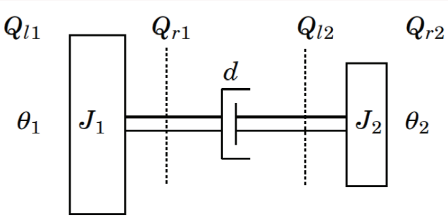
$$\begin{cases} V_1\dot{x_1} + qx_1 & = & qu \\ V_2\dot{x_2} - qx_1 + qx_2 & = & 0 \end{cases}$$

$$\dot{x} = \begin{bmatrix} -\frac{1}{V_1} & 0 \\ \frac{1}{V_2} & -\frac{1}{V_2} \end{bmatrix} qx + \begin{bmatrix} \frac{1}{V_1} \\ 0 \end{bmatrix} qu$$

If $V_1 = 0$ or $V_2 = 0$, the system becomes first order

Often simulation code, controller design methods etc have problems to treat such special cases easily

Example: Rotating Masses



$$
\begin{aligned}
J_1\dot{\omega}_1 &= Q_{l1} + Q_{r1} & \dot{\theta}_1 &= \omega_1 \\
J_2\dot{\omega}_2 &= Q_{l2} + Q_{r2} & \dot{\theta}_2 &= \omega_2 \\
Q_{r1} &= d(\omega_2 - \omega_1) & Q_{r1} &= -Q_{l2}
\end{aligned}
$$

where Q_{l1} and Q_{r2} are known time functions and J_1, J_2 and d are parameters. How is e.g. the case $J_2 = 0$ treated?

Example

General Robot Model

$$J\ddot{x}(t) + D\dot{x}(t) + Kx(t) = f(t)$$

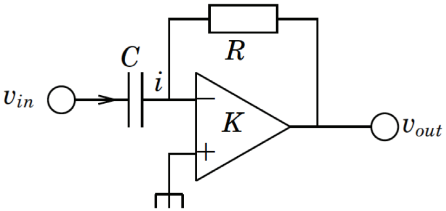
where J, D and K are matrices

Often good to use physical variables and "natural" equations

Interconnection of subsystems

How can a general system of linear differential equations be transformed, and what is the most simple form?

Example: A Differentiator



$$
\begin{aligned}
C\dot{v}_c &= i \\
\frac{1}{K} v_{out} &= -(v_{in} - v_c) \\
v_{out} &= v_{in} - v_c - Ri
\end{aligned}
$$

If $1/K = 0$, then $v_{out} = -RC\dot{v}_{in}$.

Example continued

With

$$x = \begin{bmatrix} v_c & v_{out} & i \end{bmatrix}^T$$

we have

$$sE - A = \begin{bmatrix} sC & 0 & -1 \\ 1 & -1/K & 0 \\ 1 & 1 & R \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$H = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

$$v_{out}(s) = H(sE - A)^{-1} B \, v_{in}(s) = \frac{-RCs}{\frac{RC}{K} s + \frac{K+1}{K}} \, v_{in}(s)$$

With E singular, we can describe nonproper transfer functions

General description of linear systems

Physical system described by linear differential equations, input u, output y and internal physical variables ζ

$$\begin{aligned} P(s)\zeta &= Q(s)u \\ y &= R(s)\zeta + W(s)u \end{aligned}$$

Matrix notation with **Rosenbrock system matrix**

$$\begin{pmatrix} P(s) & Q(s) \\ -R(s) & W(s) \end{pmatrix} \begin{pmatrix} -\zeta \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ y \end{pmatrix}$$

The transfer function is

$$G(s) = R(s)P^{-1}(s)Q(s) + W(s)$$

Special cases

$$\text{Right Fraction} \quad y = N_R D_R^{-1} u : \quad P = \begin{pmatrix} D_R(s) & I \\ -N_R(s) & 0 \end{pmatrix}$$

$$\text{Left Fraction} \quad y = D_L^{-1} N_L u : \quad P = \begin{pmatrix} D_L(s) & N_L(s) \\ I & 0 \end{pmatrix}$$

$$\text{State Space} : \quad P = \begin{pmatrix} sI - A & B \\ -C & D \end{pmatrix}$$

$$\text{Descriptor} : \quad P = \begin{pmatrix} sE - A & B \\ -C & D \end{pmatrix}$$

Definition: Equivalence Transformations

Two systems are “equivalent” if there are unimodular matrices $M_1(s)$, $M_2(s)$ and polynomial matrices $X(s)$ and $Y(s)$ such that

$$\begin{pmatrix} M_1(s) & 0 \\ X(s) & I \end{pmatrix} \underbrace{\begin{pmatrix} P_1(s) & Q_1(s) \\ -R_1(s) & W_1(s) \end{pmatrix}}_{P_1} \begin{pmatrix} M_2(s) & Y(s) \\ 0 & I \end{pmatrix} = \underbrace{\begin{pmatrix} P_2(s) & Q_2(s) \\ -R_2(s) & W_2(s) \end{pmatrix}}_{P_2}$$

It can be seen that this corresponds to natural transformations of variables and equations.

Fact: Any Rosenbrock system matrix is equivalent to one in state space form

$$\begin{pmatrix} P(s) & Q(s) \\ -R(s) & W(s) \end{pmatrix} \sim \begin{pmatrix} sI - A & B \\ -C & J(s) \end{pmatrix}$$

Controllability and Observability

From the transformation to state space form

$$\begin{pmatrix} M_1(s) & 0 \\ X(s) & I \end{pmatrix} \begin{pmatrix} P(s) & Q(s) \\ -R(s) & W(s) \end{pmatrix} \begin{pmatrix} M_2(s) & Y(s) \\ 0 & I \end{pmatrix} = \begin{pmatrix} sI - A & B \\ -C & J(s) \end{pmatrix}$$

we see that Smith forms are related as

$$\begin{aligned} P(s) &\sim sI - A \\ \begin{pmatrix} P(s) & Q(s) \end{pmatrix} &\sim \begin{pmatrix} sI - A & B \end{pmatrix} \\ \begin{pmatrix} P(s) \\ -R(s) \end{pmatrix} &\sim \begin{pmatrix} sI - A \\ -C \end{pmatrix} \end{aligned}$$

Controllability $\Leftrightarrow$ P, Q left coprime
Observability $\Leftrightarrow$ P, R right coprime

Irreducibility

A system

$$\mathcal{P} = \begin{pmatrix} P(s) & Q(s) \\ -R(s) & W(s) \end{pmatrix}$$

is called **irreducible** if P, Q are left coprime and P, R are right coprime

All state space descriptions equivalent to $\mathcal{P}$ are then controllable and observable, and hence minimal.

Consequence: All irreducible systems having the same transfer function are equivalent.

Poles and zeros

Transfer function on Smith-McMillan form

$$G(s) = U(s) \underbrace{\begin{bmatrix} \operatorname{diag}(\epsilon_i(s)) & 0 \\ 0 & 0 \end{bmatrix}}_{\mathcal{E}(s)} \underbrace{\begin{bmatrix} \operatorname{diag}(\psi_i(s)) & 0 \\ 0 & I_{m-r} \end{bmatrix}}_{\Psi_R(s)}^{-1} V(s)$$

$$\text{System Matrix: } \mathcal{P} = \begin{bmatrix} \Psi_R(s) & V(s) \\ -U(s)\mathcal{E}(s) & 0 \end{bmatrix} \sim \begin{bmatrix} I & 0 \\ 0 & \mathcal{E}(s) \end{bmatrix}$$

Any other irreducible system $\mathcal{P} = \begin{pmatrix} P(s) & Q(s) \\ -R(s) & W(s) \end{pmatrix}$ having the same transfer function $G(s)$ must be equivalent, therefore

The **poles** of G are given by $\det P(s) = 0$
The **zeros** of G are given by the invariant polynomials of $\mathcal{P}$

Course review

Continuous time-varying linear (CT-LTV) system

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) + D(t)u(t) \end{aligned} \tag{1}$$

Discrete time-varying linear (DT-LTV) system

$$\begin{aligned} x(k+1) &= A(k)x(k) + B(k)u(k) \\ y(k) &= C(k)x(k) + D(k)u(k) \end{aligned} \tag{2}$$

Time-domain analysis: solutions and transition matrix

Solution to CT-LTV system: with transition matrix $\Phi(t, t_0)$

$$\begin{aligned} x(t) &= \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,\sigma)B(\sigma)u(\sigma)d\sigma \\ y(t) &= C(t)\Phi(t,t_0)x_0 + \int_{t_0}^{t} C(t)\Phi(t,\sigma)B(\sigma)u(\sigma)d\sigma + D(t)u(t) \end{aligned}$$

Special cases for the transition matrix $\Phi(t, t_0)$:

- CT-LTI system: $\Phi(t, t_0) = e^{A(t-t_0)}$;
- CT-LTV system with commutative $A(t)$: If $A(t)\int_{t_0}^{t} A(\sigma)d\sigma = \int_{t_0}^{t} A(\sigma)d\sigma A(t)$ then $\Phi(t, t_0) = \text{exp}\left\{\int_{t_0}^{t} A(\sigma)d\sigma\right\}$

The AJL formula: $\det \Phi(t, t_0) = \exp\left(\int_{t_0}^{t} \mathrm{tr}[A(\sigma)]d\sigma\right)$

Time-domain analysis: stability

- For CT-LTI system: stability determined by the eigenvalues of A: A Hurwitz matrix (eigenvalues with negative real part) implies asymptotic stability;
- For CT-LTV system: stability is NOT determined by eigenvalues of $A(t)$.

Transition matrix conditions for stability $x(t)$ of $\dot{x}(t) = A(t)x(t)$:

uniformly stable if $\exists \gamma > 0$

$$\|\Phi(t,t_0)\| \leq \gamma, \quad \forall t \geq t_0 \geq 0$$

uniformly asymptotically stable if it is uniformly stable and

$$\forall \delta > 0 : \exists T > 0 :$$

$$\|\Phi(t,t_0)\| \leq \delta, \quad \forall t \geq t_0 + T, \; t_0 \geq 0$$

uniformly exponentially stable if $\exists \gamma, \lambda > 0$ such that

$$\|\Phi(t,t_0)\| \leq \gamma e^{-\lambda(t-t_0)}, \quad \forall t \geq t_0 \geq 0$$

Time-domain analysis: stability by Lyapunov function

1. There exists $\eta > 0, \rho > 0, Q(t)$:

$$\eta I \leq Q(t) \leq \rho I, \quad A^T(t)Q(t) + Q(t)A(t) + \dot{Q}(t) \leq 0$$

$\Rightarrow |x|^2 \leq \rho/\eta |x(t_0)|^2 \Rightarrow$ uniform stability

2. There exists $\eta > 0, \rho > 0, \nu > 0, Q(t)$:

$$\eta I \leq Q(t) \leq \rho I, \quad A^T(t)Q(t) + Q(t)A(t) + \dot{Q}(t) \leq -\nu I$$

$\Rightarrow |x|^2 \leq \frac{\rho}{\eta} e^{-\frac{\nu}{\rho}(t-t_0)} |x(t_0)|^2 \Rightarrow$ uniform exponential stability (equivalent to uniform asymptotic stability).

3. There exists $\rho > 0, \nu > 0, Q(t), t_0$:

$$\|Q(t)\| \leq \rho, \quad A^T(t)Q(t) + Q(t)A(t) + \dot{Q}(t) \leq -\nu I$$

$Q(t_0)$ not pos. semidef. $\Rightarrow$ not uniform stable

Under controllability and observability conditions: Uniform BIBO stability (external stability) $\Leftrightarrow$ uniform exponential stability (internal stability)

Controllability and observability

Controllability Gramian

$$W(t_0, t_f) = \int_{t_0}^{t_f} \Phi(t_0, t)B(t)B(t)^T\Phi(t_0, t)^T dt$$

The state equation is controllable on (t_0, t_f) if and only if the controllability Gramian $W(t_0, t_f)$ is invertible ($W(t_0, t_f) > 0$).

Observability Gramian:

$$M(t_0, t_f) = \int_{t_0}^{t_f} \Phi(t, t_0)^T C(t)^T C(t)\Phi(t, t_0) dt$$

The system $\dot{x}(t) = A(t)x(t), y(t) = C(t)x(t)$ is observable on (t_0, t_f) if and only if $M(t_0, t_f) > 0$.

Controllability and observability

Controllability Gramian

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The system $\dot{x}(t) = A(t)x(t), y(t) = C(t)x(t)$ is observable on (t_0, t_f) if and only if $M(t_0, t_f) > 0$.

Controllability and observability: CT-LTI systems

The following four conditions are equivalent (for controllability):

(i) The system $\dot{x}(t) = Ax(t) + Bu(t)$ is controllable.

(ii) $\mathrm{rank}[B\ AB\ A^2B\ \dots\ A^{n-1}B] = n$.

(iii) $\lambda \in \mathbf{C}$, $p^T A = \lambda p^T$, $p^T B = 0 \quad \Rightarrow p = 0$.

(iv) rank $[\lambda I - A \quad B] = n \quad \forall \lambda \in \mathbf{C}$.

The following four conditions are equivalent (for observability):

(i) The system $\dot{x}(t) = Ax(t)$, $y(t) = Cx(t)$ is observable.

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Controllability and observability: CT-LTI systems

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Realization

Conditions for realizability (time factorization from weighting pattern): The weighting pattern $G(t, \sigma)$ has a realization of dimension n if and only if there exist matrix functions $H(t) \in \mathbf{R}^{p \times n}$, $F(t) \in \mathbf{R}^{n \times m}$ such that $G(t, \sigma) = H(t)F(\sigma) \quad \forall t, \sigma$.

Conditions for minimal realisation: the realized linear system is controllable and observable.

Algorithms for realization: Gilbert realization (partial fraction expansion of transfer functions), Markov parameters etc.

Least squares and minimum energy control

Least squares problem I: Minimize $|Lu - v|$ with respect to u.
Solution: Any $\hat{u}$ satisfying the Orthogonality Property
$0 =< Lx, L\hat{u} - v >$ for all x.
Or equivalently

$$L^* L\hat{u} \quad = \quad L^* v$$

Application: estimating initial state from LTV (LTI) system by output measurement (under observability condition).

Least squares problem II: Minimize $|u|$ under the constraint $Lu = v$.
Solution: Any $\hat{u}$ satisfying $L\hat{u} = v$ and the Orthogonality Property
$0 =< \hat{u}, \hat{u} - u >$ for all u with $Lu = v$.
Or, if LL^* invertible, equivalently

$$\hat{u} \quad = \quad L^*(LL^*)^{-1}v \quad \text{(if } LL^* \text{ invertible)}$$

Application: minimum-energy control for LTV (LTI) system with boundary conditions (under controllability condition).

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Frequency-domain analysis: polynomial matrices

Polynomial matrix fraction descriptions (MFD) for MIMO transfer functions:

Right polyomial MFD: $G(s) = N_R(s)D_R(s)^{-1}$.
Left polynomial MFD: $G(s) = D_L(s)^{-1}N_L(s)$.

Coprime MFDs: unique up to unimodular matrix transformations:
For two coprime right MFDs $G(s) = N_1(s)D_1^{-1}(s) = N_2(s)D_2^{-1}(s)$ then there is a unimodular matrix $U(s)$ such that

$$N_1(s) = N_2(s)U(s), \qquad D_1(s) = D_2(s)U(s)$$

The left MFD $(sI - A)^{-1}B$ is coprime $\Leftrightarrow$ $\{A, B\}$ is controllable.

The right MFD $C(sI - A)^{-1}$ is coprime $\Leftrightarrow$ $\{A, C\}$ is observable.

Frequency-domain analysis: polynomial matrices

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Frequency-domain analysis: polynomial matrices

Zeros and poles from MIMO transfer functions:
The Smith McMillan form

$$G(s) = P(s) \begin{pmatrix} \operatorname{diag}\left(\frac{\epsilon_i(s)}{\psi_i(s)}\right) & 0 \\ 0 & 0 \end{pmatrix} Q(s)$$

where P, Q are unimodular matrices and ϵ_i, ψ_i are without common factors.

Using the Smith McMillan form one can determine

- The roots of $\epsilon_i(s)$ as the system (transmission) zeros
- The roots of $\psi_i(s)$ as the system poles

(counted with multiplicities)

Other topics

Some topics that we do not cover in the course

- Feedback control (state feedback or output feedback)
- State observation
- LQR/LQG optimal control
- Geometric theory in linear system

You will find them in the two textbooks (Rugh and Hespanha).

Final exam

Problems in the final exam will be confined to those presented in the lecture slides.

Skip the following topics from lecture slides

- Time-varying transfer functions (for LTV/LTP systems), Lecture 2;
- Balanced realizations and bonus contents, Lecture 3;
- Feedback, well-posedness (for internal stability), Lecture 5;
- Polynomial interpolation/function approximation with LS methods, Lecture 6.

Final exam will be a 24-hour take-home exam. Date to be determined.

THE END