Automatic Control – Basic Course

Laboratory Exercise 2

Process Modelling and PID Controller Design

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Practical Things

You log in using the account name `lab_tanka` (just as in lab 1). Leave the password field empty. The two windows of the graphical user interface are opened automatically at login together with MATLAB, running in console mode.

1. Introduction

Laboration 1 gave practical experience of, and insight into, PID control. However, the laboration lacked a systematic method for choosing the controller parameters. The objective of this laboration is to show how one can construct a mathematical model of the process to be controlled and choose suitable controller parameters using this model.

The laboration is carried out on the same tank system, which was used during laboration 1, see Figure 1.
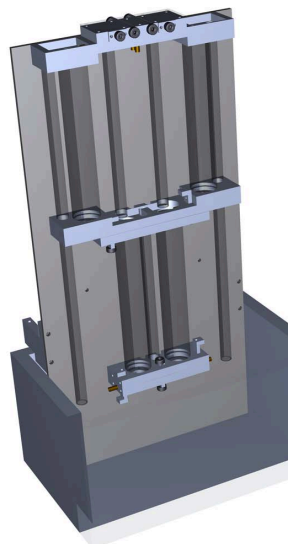

Figure 1   Lab setup (of which the left half is used).

Preparations

In order to get as much as possible out of the lab, you need to master the concepts of linearization, transfer function, characteristic polynomial and pole placement.

You shall have read through this manual. You shall also have worked through the preparatory exercises 2.1, 2.2, 2.3, 2.4, 2.5, 2.7, 2.8, 3.1 and 3.6. Co-operation is allowed (and endorsed). Observe that assignments 3.1 and 3.6 are completed during class. Re-read the appendix of laboration 1 (on the user interface) if you have forgotten any details.

The laboration starts with a written test, consisting of two randomly chosen review questions. **The answer of both these questions must be reasonably correct in order for you to continue with the laboration. Additionally, you shall be able to account for your solutions to the preparatory exercises, if the lab assistant asks you.** The review questions are found on page 19.

Don't forget to bring the manual from lab 1 in order to compare results. (Also, the appendices describing the lab interface are found in the manual of lab 1.)

2. Modelling

In this section we will derive a mathematical model of the tank system, originating in physical principles and construction data. The obtained mathematical model shall then be verified through experiments.

Prior to the actual modelling, it might be worthwhile to pay attention to a practical aspect, which is uncomplicated, but nevertheless a frequent cause of control system implementation errors.

Units and Unit Conversion In this (and the previous) laboration a physical process is controlled using a controller. In physical reality we deal with signals of diverse units, such as levels [m], flows [$m^3 s^{-1}$] and voltages [V]. Our controller is implemented in a digital computer, which can only handle unit-less numbers.

Units of constants and signals will be explicitly given, in order not to get lost in the 'unit jungle'. We use [1] to denote a unit-less entity. At first sight, this may seem superfluous. However, the absence of a proper unit analysis easily leads to errors which are both severe and hard to locate.

The block diagram in figure 2 illustrates our closed loop control system. The in- and output signals of the controller are unit-less and normalized to the interval $(0, 1)$. The control signal is converted to a voltage by means of a D/A (Digital/Analog) converter. The voltage signal is amplified and used to drive the pump. Here, we consider the amplifier to be a internal part of the pump and the pump voltage to be linearly proportional to flow through the pump.

The water level in either tank is measured using pressure sensors, generating a voltage proportional to the water pressure (i.e., proportional to the level). This voltage is converted to a unit-less number, normalized to $(.0, 1)$ by means of an A/D (Analog/Digital) converter.

It is not obvious where to place the division between process and controller. However, it is practise to place it so that the in- and output of the process obtain the same unit. A natural way to achieve this is to consider the D/A and A/D converts as part of the
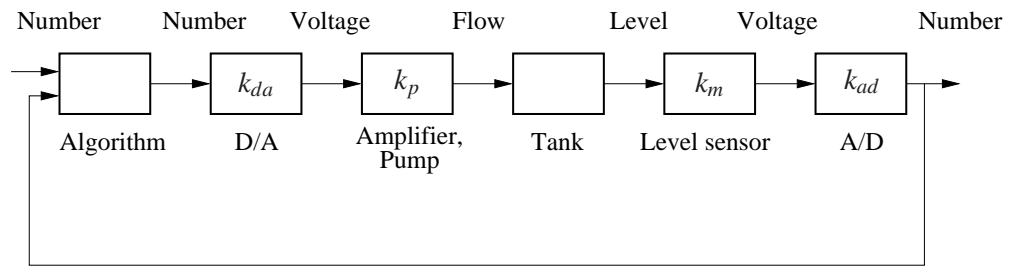
Figure 2 Block diagram of the control system with unit converters k_{da}, k_p, k_m and k_{ad}.

process, rather than the controller. Hence, the in-and outputs of the controller become unit-less signals normalized to $(0, 1)$.

In order to simplify our work, the block diagram in Figure 2 is replaced by the equivalent scheme of Figure 3.

Assignment 2.1 (Preparation) What are the units of the conversion factors k_{da}, k_p, k_m and k_{ad} in Figure 2? Express the new constants α and β in Figure 3 in the constants of Figure 2. Especially, give the units of the new constants.
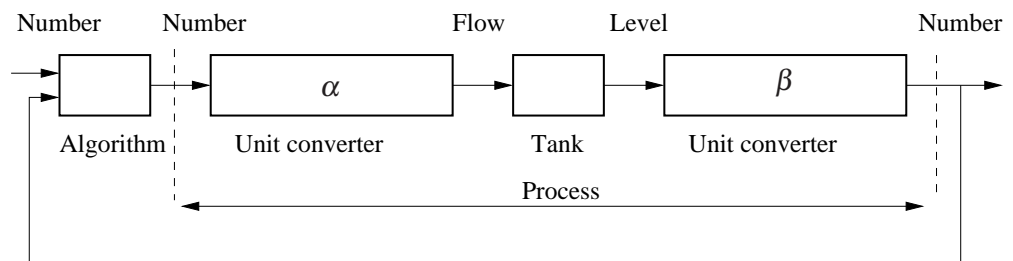

Figure 3 Equivalent block diagram for the closed loop system.

We have now completed our unit analysis and obtained a process model with unit-less in- and outputs, in the interval $(0, 1)$. The interval $(0, 1)$ corresponds to (pump turned off, pump running at max) for the input and (empty tank, full tank) for the output.

After this important, and unfortunately often neglected, step we are ready to assemble a mathematical model of our control system, which will be used to design various controllers.

Assignment 2.2 (Preparation) Use volume balance to derive a differential equation describing the relation between the inflow $q_u(t)$ [m^3 s^{-1}] to the upper tank and the heights $h_1(t)$ [m] and $h_2(t)$ [m] of the upper and lower tanks, respectively. Within each tank, an approximate relation between outflow speed $v(t)$ [m s^{-1}] and water level $h(t)$ [m] is given by Torricelli's law:

$$v(t) = \sqrt{2gh(t)}$$

3

where $g = 9.81$ [m s^{-2}] is the acceleration of gravity. Denote by A_1 [m^2] and A_2 [m^2] the cross section areas of the upper and lower tank, respectively. Further, let a_1 [m^2] and a_2 [m^2] denote the corresponding outflow cross sections. Neglect any dynamics within the hoses and the pump.

Hint: When a liquid with speed $v(t)$ [m s^{-1}] flows through a hole of cross section a [m^2], the flow is given by $q(t) = a\,v(t)$ [m^3 s^{-1}]. See Figure 3 and be consistent with the usage of units.

We now have a physical model with flow [m^3 s^{-1}] as input and levels [m] as states (outputs), while the in- and outputs of the desired model are unit-less numbers, normalized to $(0,1)$.

Assignment 2.3 (Preparation)   Introduce the unit conversions $q_u = \alpha u$, $h_1 = x_1/\beta$ and $h_2 = x_2/\beta$ in your model from assignment 2.2. Let the state variables x_1, x_2 [1] be the tank levels h_1, h_2 [m], normalized to $(0,1)$.

Assignment 2.4 (Preparation)   The tanks in this lab have the same cross section areas, i.e., $A = A_1 = A_2$ [m^2]. Show that the process is described by the model

$$\frac{dx_1(t)}{dt} = -\gamma_1\sqrt{x_1(t)} + \delta u(t)$$
$$\frac{dx_2(t)}{dt} = \gamma_1\sqrt{x_1(t)} - \gamma_2\sqrt{x_2(t)} \tag{1}$$

where

$$\gamma_1 = \frac{a_1}{A}\sqrt{2g\beta}\ [s^{-1}], \quad \gamma_2 = \frac{a_2}{A}\sqrt{2g\beta}\ [s^{-1}], \quad \delta = \frac{\alpha\beta}{A}\ [s^{-1}]$$

Note: This is a state space description of a non-linear system. How can this be seen? The interval $(0,1)$ [1] in the states x_1, x_2 corresponds to the water level interval $(0, 0.16)$ [m] in the upper and lower tank, respectively. Compute theoretical values for the parameters δ, γ_1 and γ_2 from the construction data shown below. Insert the results in the table below.

Tank cross sections:	$A_1 = A_2 = 4.9 \cdot 10^{-4}$ m^2
Outlet cross sections:	$a_1 = a_2 = 3.1 \cdot 10^{-6}$ m^2
Conversion factor from control to flow:	$\alpha = 2.1 \cdot 10^{-5}$ m^{3}s^{-1}
Conversion factor from height to measurement:	$\beta = 6.25$ m^{-1}

Assignment 2.5 (Preparation)   In practice, there exist differences between the individual tank processes. Additionally, their characteristics vary with time – the holes get congested, the pump gets worn, etc. Hence, the theoretical values are not absolutely reliable. However, the actual values can be estimated by means of simple experiments:

- δ can be estimated by blocking the outflow of the upper tank (by means of AV3), issuing a constant control signal and measuring the time it takes for the measurement signal to increase from a given value to another.

- γ_1 and γ_2 can be estimated by issuing a constant control signal, waiting until the system reaches equilibrium and then reading the corresponding stationary states x_1^0 and x_2^0.

Show, starting with equation (1), how the above experiments can be used to compute experimental values for first δ, then γ_1 and finally γ_2.

Assignment 2.6   Log in according to the description in the beginning of this manual (if you have not already done so). Conduct the experiments and calculations in accordance with assignment 2.5 in order to determine experimental values of δ, γ_1 and γ_2.

	Theoretical value	Experimental value
δ		
γ_1		
γ_2		

Confirm that the experimental values match the theoretical ones to some extend.

Preferably, base your controller design on the experimental values. Why?

Note: We are now done with modelling and identification of process parameters. The obtained model has normalized and unit-less in- and outputs as well as states. Consequently, the notion of units can be dropped from here on.

Assignment 2.7 (Preparation)   Linearize the system (1) around an arbitrary equilibrium (x_1^0, x_2^0). (In the laboration, $x_1^0 = 0.5$ and $x_2^0 = 0.5$ will be used.)

Assignment 2.8 (Preparation)   Show that the linearized system from assignment 2.7 can be described by the following transfer function, where the output is given by $y_1(t) = x_1(t)$, $y_2(t) = x_2(t)$.

$$
\begin{aligned}
\Delta Y_1(s) &= \frac{\rho \tau_1}{1 + s\tau_1} \Delta U(s) \\
\Delta Y_2(s) &= \frac{\rho \tau_2}{(1 + s\tau_1)(1 + s\tau_2)} \Delta U(s)
\end{aligned}
\tag{2}
$$

Determine the parameters ρ, τ_1 and τ_2 as functions of the process parameters δ, γ_1 and γ_2 at the operating point x_1^0, x_2^0.

3. Computation of Controller Parameters

In this section, we shall compute controller parameters for controlling the upper and lower tank, respectively. We start out with the mathematical models, which were obtained in the previous section. The controllers will eventually be evaluated on the real system.

The controller parameters will depend on specifications, which we wish the controlled system to meet. A specification can be of several natures; here we will consider the poles of the closed loop system. As you know, the pole placement has an intuitive interpretation in terms of speed and damping of the closed loop system.

In this laboration, we will work with PI and PID controllers. By choosing the PI(D) parameters in an appropriate way, a desired characteristic polynomial (denominator polynomial) of the closed loop system can be obtained. See Figure 4.
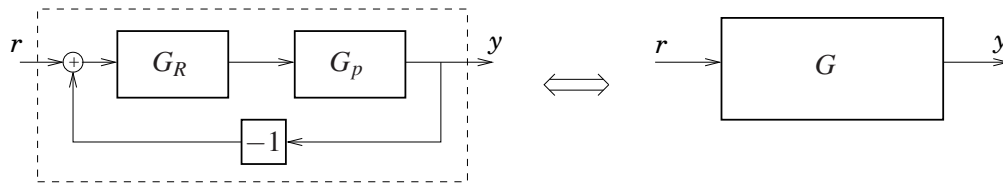

Figure 4 The closed loop system is specified through a desired characteristic polynomial.

The process transfer function $G_p(s)$ and the controller transfer function $G_R(s)$ are given by

$$Y(s) = G_P(s)U(s)$$
$$U(s) = G_R(s)E(s) = G_R(s)\left(R(s) - Y(s)\right) \tag{3}$$

From these transfer functions, the closed-loop transfer function becomes

$$Y(s) = G(s)R(s) = \frac{G_P(s)G_R(s)}{1 + G_P(s)G_R(s)}R(s) \tag{4}$$

The goal is to determine the parameters of controller $G_R(s)$ so that the closed-loop transfer function $G(s)$ gets a desired denominator polynomial.

Control of the Upper Tank

Assignment 3.1 (Preparation)   Show that the closed-loop transfer function G is determined by process transfer function G_p and controller transfer function G_R according to Equation 4. The process transfer function of the upper tank is given by the first equation of (2) from assignment 2.8. The controller transfer function of a PI controller is given by

$$u(t) = K\left(e(t) + \frac{1}{T_i}\int_0^t e(\tau)d\tau\right) \quad \Leftrightarrow \quad U(s) = K\left(1 + \frac{1}{sT_i}\right)E(s) \tag{5}$$

Use these two transfer functions to design a PI controller for the upper tank. Choose the controller parameters so that the closed loop system obtains a relative damping ζ and an undamped angular frequency of oscillation ω, i.e., the characteristic polynomial of the closed loop system should be

$$s^2 + 2\zeta\omega s + \omega^2 \tag{6}$$

The controller parameters K and T_i shall be expressed in the process parameters ρ and τ_1 together with the design parameters ω and ζ.

Assignment 3.2   We start by designing a controller with $\zeta = 1, \omega = 0.3$ for the upper tank, with the operation point chosen to $x_1^0 = 0.5$. This can be done by means of the MATLAB-script `calcpi` according to the following example (insert your estimated values of `delta`, `gamma1` and `gamma2`):

```
>> delta = ... ;
>> gamma1 = ... ;
>> gamma2 = ... ;
>> omega = 0.3;
>> zeta = 1;
>> calcpi
K =
    2.1964
Ti =
   6.1010
```

Have a look at what the script does by typing

```
>> type calcpi
```

and compare the calculations with those you made as part of the preparations.

Assignment 3.3   We shall now evaluate the control performance at reference and load steps. A suggested experimental procedure is the following:

1. Ensure the interface is set to PI control of the upper tank, i.e., that `Control Mode` shows `Automatic`, `Tank Selection` shows `Upper` and that the P and I blocks are active (white) while the D block is inactive (light blue).

2. Ensure that the valves `AV1`, `AV2`, `BV2`, `BV1` are pressed in and that the valves `AV3`, `BV3`, `AV4`, `BV4`, `V5` are pressed down.

3. Adjust the controller parameters `K` and `Ti`.

4. Set the reference `r` to 0.5 and wait until all signals have reached stationarity.

5. Conduct a set point change (step) to 0.7 and depict the response in the figure below.

6. Reset the reference to 0.5.

7. When the system has reached stationarity anew, open the load disturbance valve `BV1` and depict the resulting response in the same figure.

Hint: Your results should show the same characteristics as seen in figure 5. If you feel uncertain, talk with the lab assistant prior to moving on with the lab.
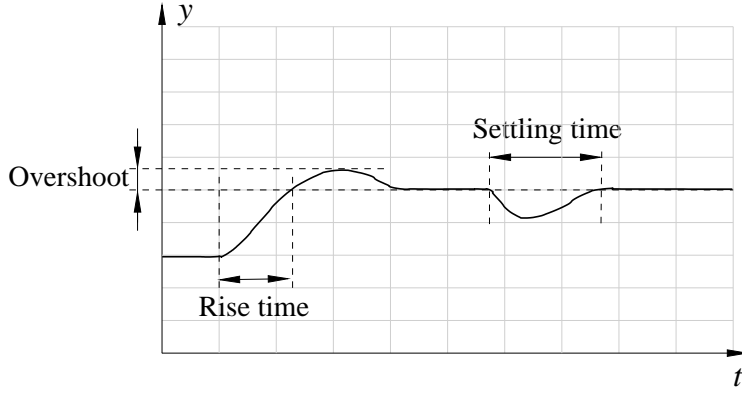
Figure 5 Definition of rise time and overshoot at reference steps and settling time at load steps, respectively.
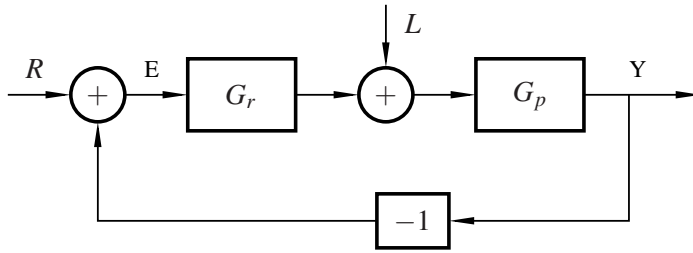

Figure 6 Block diagram of the closed loop system during control of the upper tank.

Poles and Zeros The system, under closed loop control of the upper tank, is described by the block diagram in Figure 6. The closed loop system transfer functions from reference $R(s)$ and load disturbance $L(s)$ to output $Y(s)$ are given by

$$G_{YR}(s) = \frac{G_p(s)G_r(s)}{1 + G_p(s)G_r(s)} = \frac{\rho K(s + \frac{1}{T_i})}{s^2 + s(\frac{1}{\tau_1} + \rho K) + \frac{\rho K}{T_i}} \qquad (7)$$

$$G_{YL} = \frac{1}{1 + G_p(s)G_r(s)} = \frac{s\rho}{s^2 + s(\frac{1}{\tau_1} + \rho K) + \frac{\rho K}{T_i}} \qquad (8)$$

We have designed our controllers using pole placement, i.e., by choosing the characteristic polynomial of the closed loop system. No consideration has been taken to how the controller affects the zeros of the closed loop system. The zeros of the transfer function G_{YR}, from reference to measurement, are moved when controller parameters are changes. The zeros of the system G_{YL} are, however, unaffected by the controller parameters. This explains why reference steps (but not load disturbance steps) may generate an overshoot in the measurement signal, despite a relative damping of $\zeta = 1$. In order to investigate how the location of poles affect the behavior of the system, we should hence primarily study load responses. If we wish to investigate the combination of pole and zero dynamics, we may study the response to reference changes. A closer study of zeros is made during lecture 12.
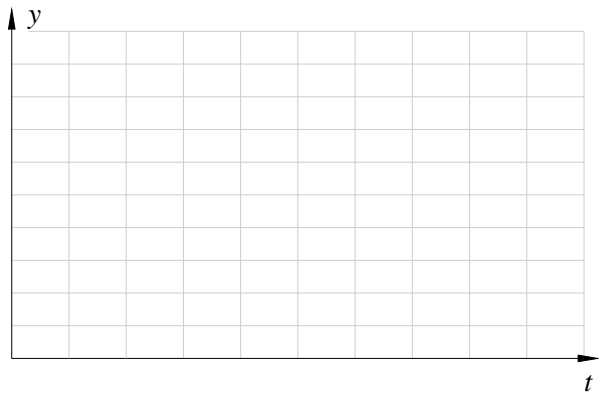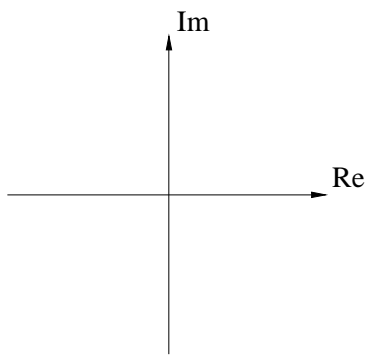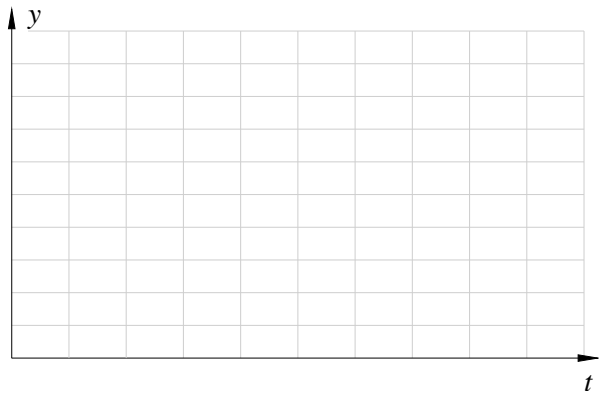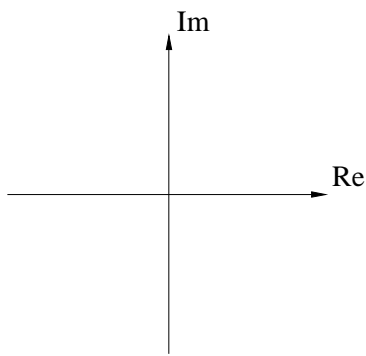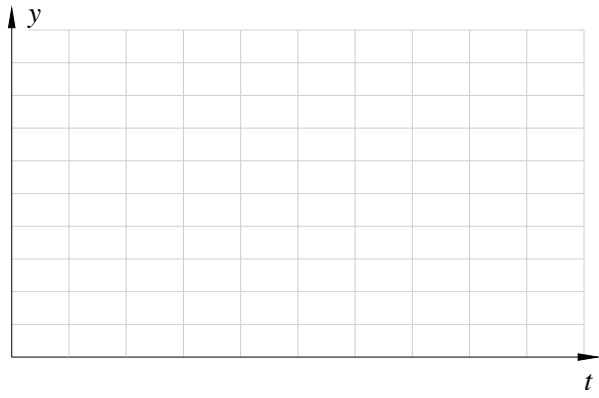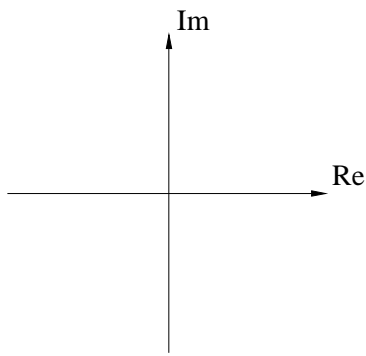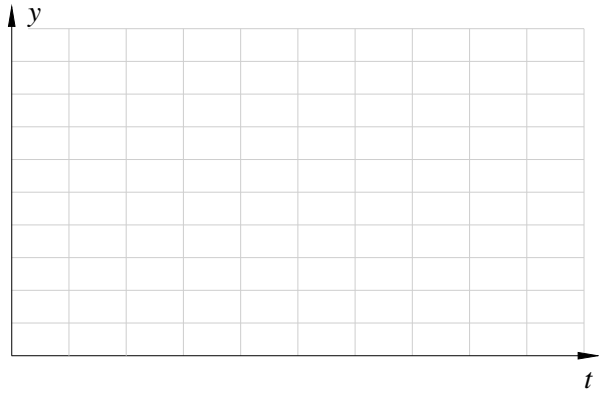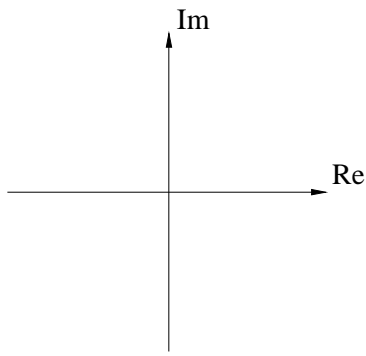
Assignment 3.4 Now that we have investigated the difference between reference and load steps, we are ready to investigate how the relative damping ζ and natural frequency ω affect the characteristics of the closed loop system.

First we fix ζ to 1 and vary ω according to the table below. Assume that the stationary level is given by $x_1^0 = 0.5$ and compute PI parameters K och T_i for every choice of ω.

Conduct the experiment according to the instructions of assignment 3.3. Complete the table, singularity charts and time plots below. If the control signal reaches one of its limits, it becomes saturated. Note in the table if if this happens (yes/no).

Finally, try the controller you arrived at during laboration 1 (and complete the final row of the table).

				Reference change			Load disturbance.
ω	ζ	K	T_i	Rise time [s]	Overshoot	Saturation	Settling time [s]
0.1	1						
0.2	1						
0.5	1						
X	X						

Assignment 3.5   Fix ω to 0.4 and vary ζ according to the table below. Compute the controller parameters K and T_i by means of MATLAB in the same way as before. Try the controller parameters on the upper tank and investigate the responses to reference changes and load disturbances. Depict the responses in the time plots below. Also depict the pole locations in the singularity charts and compare them to the characteristics of the responses; especially study the damping.

Conduct the experiments in the same way as in the previous assignment.

				Reference change			Load disturbance
ω	ζ	K	T_i	Rise time [s]	Overshoot	Saturation	Settling time [s]
0.4	0.7						
0.4	0.4						
0.4	0.2						
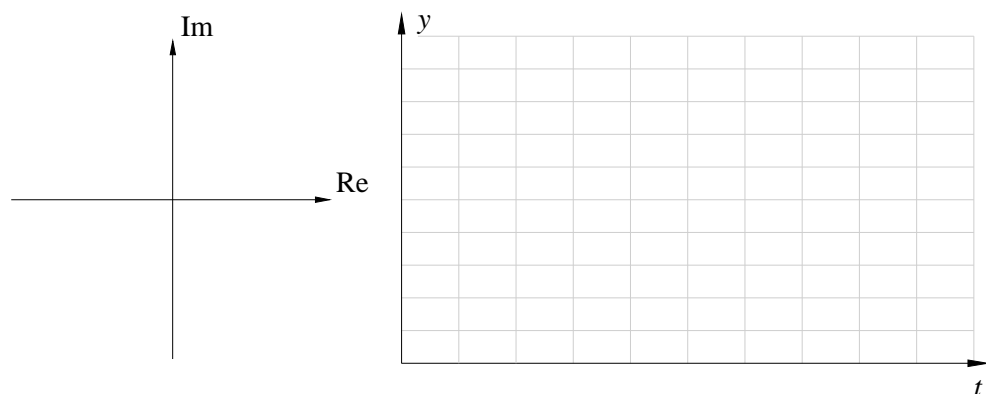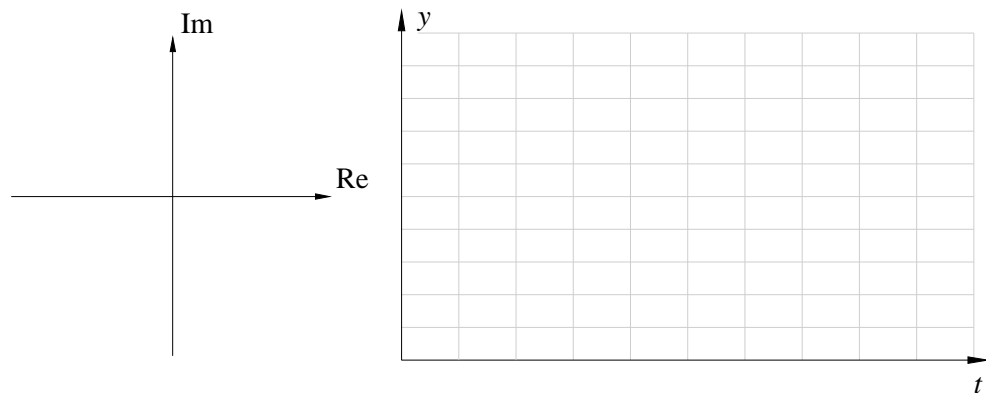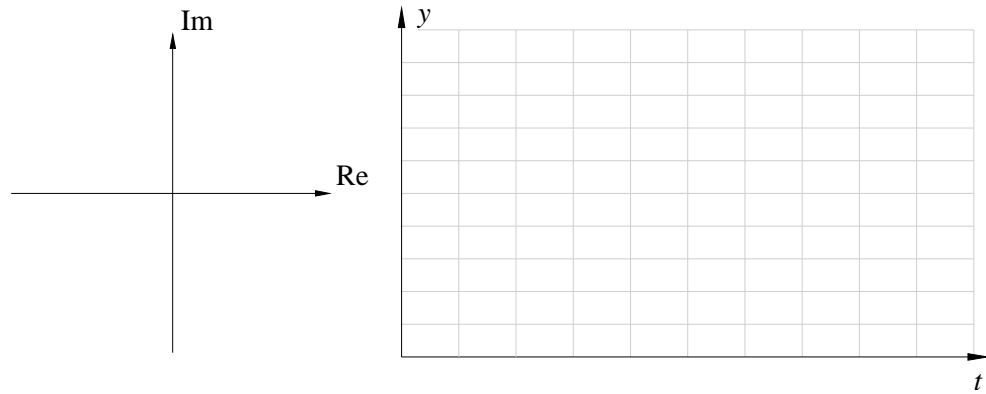
Control of the Lower Tank

Assignment 3.6 (Preparation)   Use the second equation of (2) from assignment 2.8 to design a PID controller,

$$u(t) = K\left(e(t) + \frac{1}{T_i}\int_0^t e(\tau)d\tau + T_d\frac{de(t)}{dt}\right) \quad\Leftrightarrow\quad U(s) = K\left(1 + \frac{1}{sT_i} + sT_d\right)E(s) \tag{9}$$

for control of the level in the lower tank. Choose the controller parameters such that the characteristic polynomial of the closed loop system becomes

$$(s + \alpha\omega)(s^2 + 2\zeta\omega s + \omega^2) \tag{10}$$

The controller parameters K, T_i and T_d shall be expressed in the process parameters ρ, τ_1 and τ_2 together with the design parameters ω, ζ and α.

Poles and zeros   We shall now investigate control of the lower tank. Figure 7 shows a block diagram of the closed loop system. Note the entry points of load disturbances L_1 and L_2.

The transfer functions from refernce to output (G_{YR}) as well as those from load to output (G_{YL_1}, G_{YL_2}) are given below.

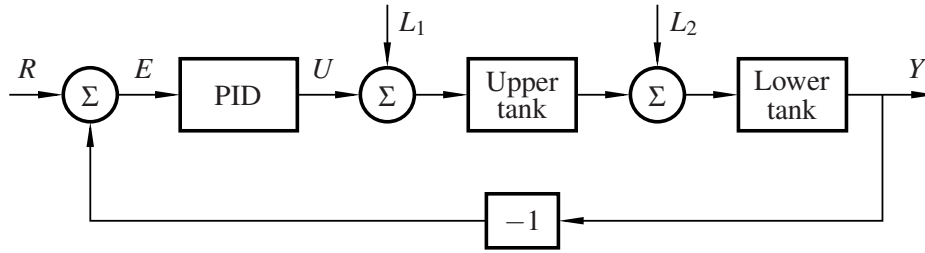**Figure 7**   Block diagram of the lower tank control system.

$$G_{YR} = \frac{K\rho(s^2\frac{T_d}{\tau_1} + s\frac{1}{\tau_1} + \frac{1}{T_i\tau_1})}{s^3 + s^2(\frac{1}{\tau_1} + \frac{1}{\tau_2} + \frac{\rho K T_d}{\tau_1}) + s(\frac{1}{\tau_1\tau_2} + \frac{\rho K}{\tau_1}) + \frac{\rho K}{T_i\tau_1}} \tag{11}$$

$$G_{YL_1} = \frac{s\frac{\rho}{\tau_1}}{s^3 + s^2(\frac{1}{\tau_1} + \frac{1}{\tau_2} + \frac{\rho K T_d}{\tau_1}) + s(\frac{1}{\tau_1\tau_2} + \frac{\rho K}{\tau_1}) + \frac{\rho K}{T_i\tau_1}} \tag{12}$$

$$G_{YL_2} = \frac{s\frac{1}{\tau_1}(s + \frac{1}{\tau_1})}{s^3 + s^2(\frac{1}{\tau_1} + \frac{1}{\tau_2} + \frac{\rho K T_d}{\tau_1}) + s(\frac{1}{\tau_1\tau_2} + \frac{\rho K}{\tau_1}) + \frac{\rho K}{T_i\tau_1}} \tag{13}$$

A load step L_1 can be generated by opening the valve BV1, while opening AV4 generates a load step L_2.

The three transfer functions have the same denominator polynomials while the numerators differ. If the controller parameters change, the closed loop poles are moved. In addition, the zeros of G_{YR} (reference to output) will be moved while zeros of G_{YL_1} and G_{YL_2} are unaffected.

In order to investigate how the location of poles affect the behavior of the system, we should hence primarily study load responses. If we wish to investigate the combination of pole and zero dynamics, we may study the response to reference changes.

Assignment 3.7   Fix ζ to 0.7, α to 1 and vary ω according to the table on the next page. Assume that the stationary level is given by $x_2^0 = 0.5$ and compute PID parameters using the MATLAB-script `calcpid` according to the following example:

```
>> omega = 0.15;
>> zeta = 0.7;
>> alpha = 1;
>> calcpid
K =
    4.0390
Ti =
   15.2320
Td =
    5.0221
```

Have a look at the script by typing

```
>> type calcpid
```

and compare the computations with those you made as part of the lab preparations.

Try the controllers on the lower tank and investigate the responses to reference changes and load disturbances generated by means of the valve BV1. Depict the responses in the time plots below. Also depict the pole locations in the singularity charts and compare them to the characteristics of the responses; especially study their speed.

Conduct the experiments the following way:

1. Ensure that the interface is set to PID control of the lower tank, i.e., that Control Mode shows Automatic, Tank Selection shows Lower and that the blocks P, I and D are active (white).

2. Make sure that the valves AV1, AV2, BV2, BV1 are pressed in, while the valves AV3, BV3, AV4, BV4, V5 are pressed down.

3. Set the controller parameters K, Ti and Td.

4. Set the reference r to 0.5 and wait until all signals have reached stationarity. The Optimal button can be used to reach the stationary point corresponding to $x_2 = r$ in minimal time.

5. Conduct a reference change (step) to 0.7 and depict the response. Write down the rise time and overshoot in the table below. Also note whether the control signal saturated and if so, during how long time.

6. When the system reaches stationarity anew, open the load disturbance valve BV1 and depict the resulting response. Write down the settling time in the table.

Finally, try the controller you arrived at during lab 1 (and complete the last row of the table).

Hint: These experiments take a fair amount of time to conduct. You may work with the summary in chapter 4 while you wait. Use the Optimal button to save time.

ω	ζ	α	K	T_i	T_d	Reference change			Load disturbance
						Rise time [s]	Overshoot	Saturation	Settling time [s]
0.1	0.7	1							
0.15	0.7	1							
0.2	0.7	1							
X	X	X							

Im

Re

y

t

Im

Re

y

t

Im

Re

y

t

Im

Re

y

t
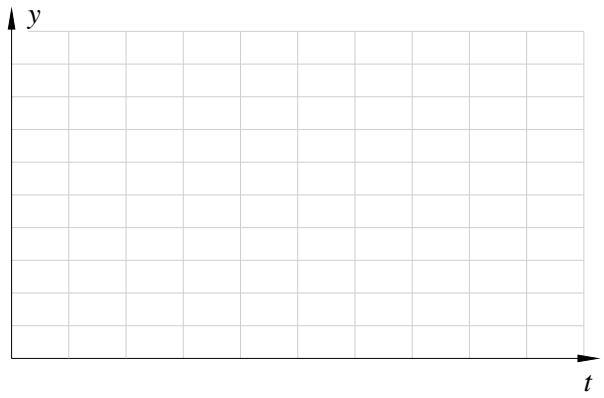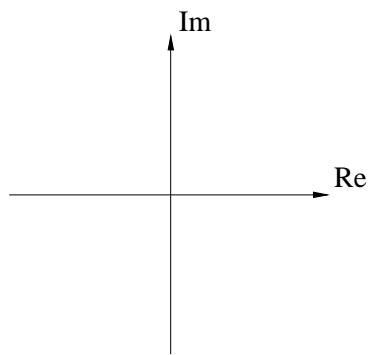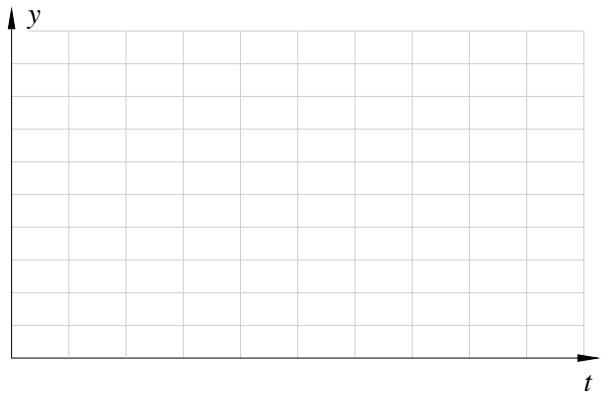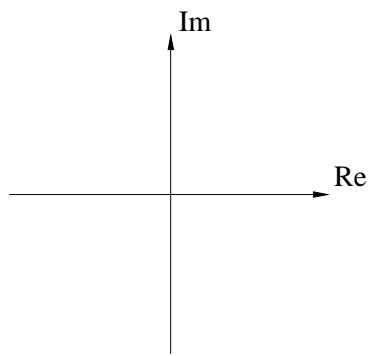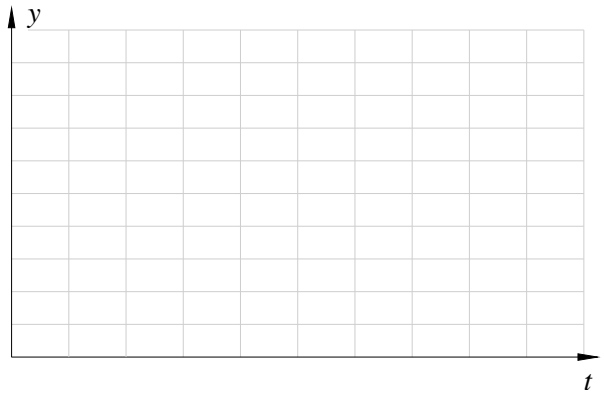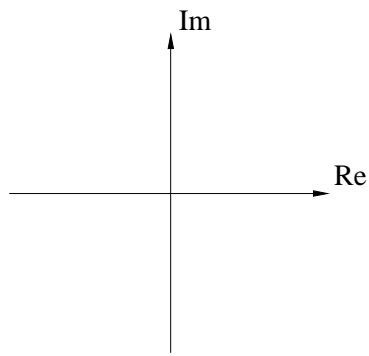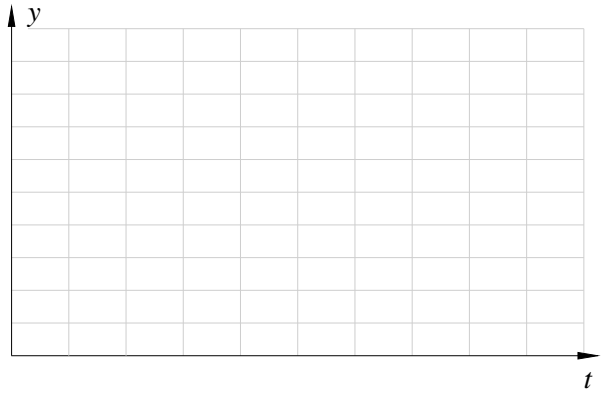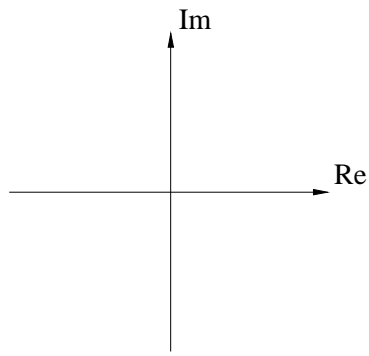
4. Summary

This summary is intended to review the methodology we have used and to collect relevant questions, which you should be able to answer once you have completed the experiments. The lab assistant will go through the summary before you may pass the lab.

Assignment 4.1   Enter the steps you have gone through before and during the lab in the empty boxes of the figure below. They should be chronologically ordered. (Observe that parameter estimation is omitted. Where would it fit in?)

- Closed loop transfer functions $Y(s) = \frac{G_p G_r}{1 + G_p G_r} R(s)$

- Physical modelling

- Nonlinear differential equation
  $\dot{x} = f(x, u)$

- Linear differential equation
  $\dot{x} = ax + bu$

- Linearization

- Laplace transform

- Pole placement specification
  $s^2 + 2\zeta\omega s + \omega^2 = 0$

- Testing

- Evaluation

- Controller parameter expressions
  $K = \ldots,\ T_i = \ldots,$

- Transfer function
  $Y(s) = G(s)U(s)$

Physical
double tank
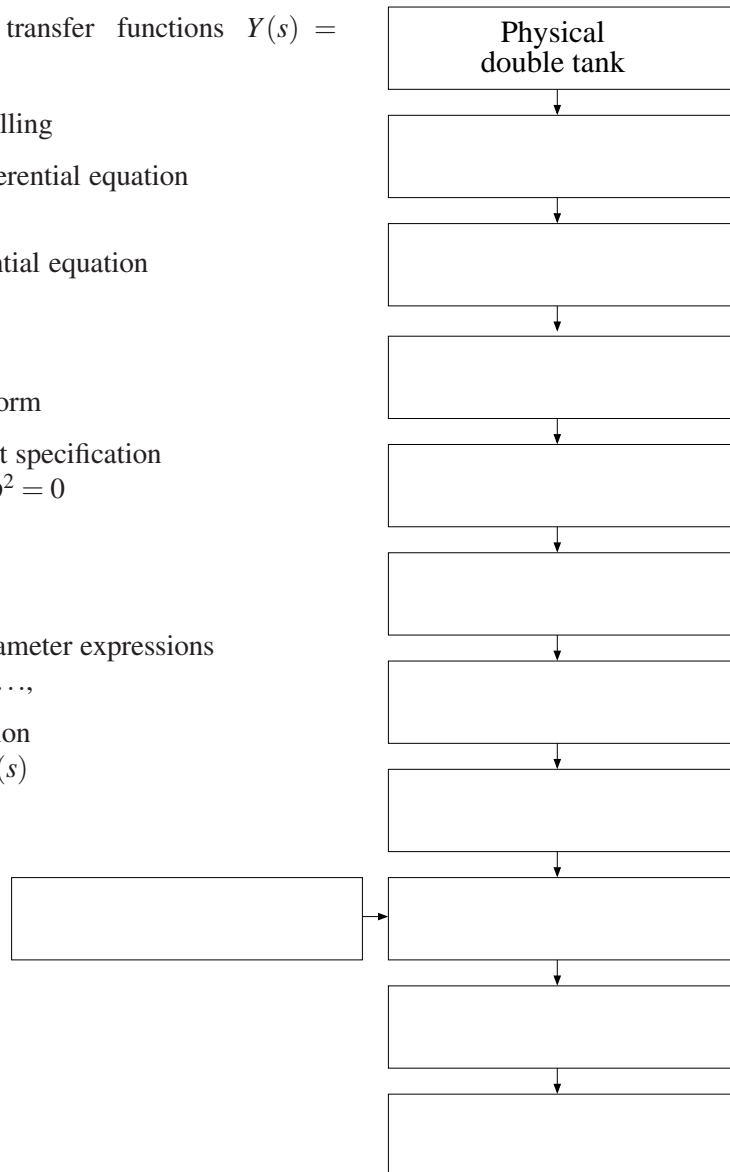
Assignment 4.2   Mention at least two limitations of the physical process, which are not captured by the mathematical model (1).

Assignment 4.3   Consider PI control of the upper tank. How do the closed loop system poles move when ω is increased? How does this affect the response to reference changes and load disturbances?

_____

_____

_____

How does K and T_i change when ω is increased? Why do we not use $\omega = 10$ rad s^{-1}?

_____

_____

_____

Assignment 4.4   Still consider PI control of the upper tank. How do the closed loop system poles move when ζ is decreased? How does this affect the response to reference changes and load disturbances? How would the step response look if we chose $\zeta = 0$?

_____

_____

_____

_____

Assignment 4.5   Why did we not use the D part for control of the upper tank?

_____

_____

Assignment 4.6   How many closed loop poles do we have when controlling the lower tank with a PID controller?

_____

_____

How do the closed loop system poles move when ω is increased? How does this affect the response to reference changes and load disturbances?

_____

_____

_____

How are K, T_i and T_d affected when ω is increased? Why do we not try $\omega = 1$ rad/s?

_____

_____

_____

Assignment 4.7   Write down your recommendations for controller parameters. Use the table below. Compare these with the parameters you arrived at during lab 1.

	övre tank	undre tank
P	$K =$	$K =$
PI	$K =$ $T_i =$	$K =$ $T_i =$
PID	$K =$ $T_i =$ $T_d =$	$K =$ $T_i =$ $T_d =$

Knowledge Review for Lab 2

1. Determine all stationary points (x^0, u^0, y^0) of the system

$$\frac{dx}{dt} = -a\sqrt{x} + bu$$
$$y = cx$$

2. Linearize the system

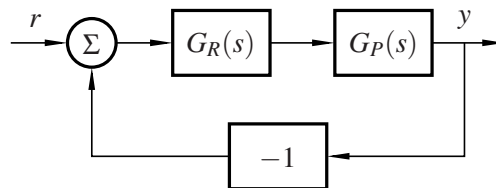$$\frac{dx}{dt} = -a\sqrt{x} + bu$$
$$y = cx$$

   around the stationary point (x^0, u^0, y^0).

3. Write down the transfer function of a

   (a) P controller

   (b) PI controller

   (c) PID controller

4. Determine the closed loop transfer function for the interconnection

   where $G_R(s) = K$ and $G_P(s) = \dfrac{1}{1+sT}$

5. When characterizing second order systems, the two parameters

   $$\zeta \qquad \text{(relative damping)}$$
   $$\omega \qquad \text{(natural frequency)}$$

   are often used. Depict how these parameters relate to the placement of system poles in a singularity chart.

6. The transfer function of a system is given by

   $$G(s) = K\frac{Q(s)}{P(s)}$$

   Consider the singularity chart to the right and determine $Q(s)$ and $P(s)$.