# Feedback – An Example, Stability, Stationary Errors

Automatic Control, Basic Course, Lecture 4

November 14, 2018

Lund University, Department of Automatic Control

# Content

# Feedback – The Steam Engine

# Control in the Old Days

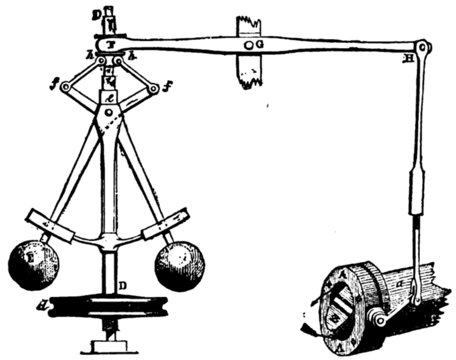
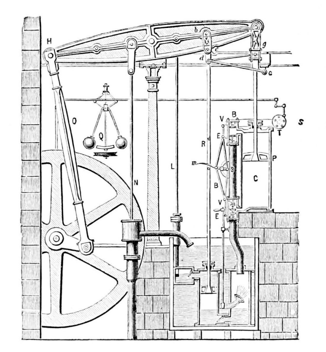
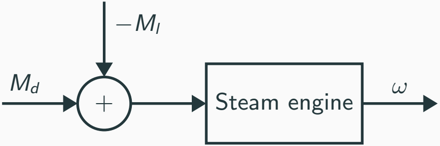



FIG. 4.—*Governor and Throttle-Valve.*




**James Watt (1788): centrifugal/fly-ball governor** for steam engines (based on Huygens work on windmills).
Watt earlier improved performance for steam engines with “condenser”.

# The Uncontrolled Steam Engine




Model:

$$J\dot{\omega} + D\omega = M_d - M_l$$

The stationary angular speed:

$$\omega_s = \frac{M_d - M_l}{D}$$

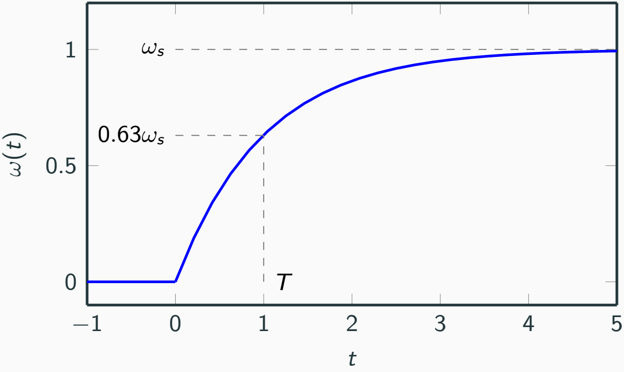
# The Uncontrolled Steam Engine

Step response:

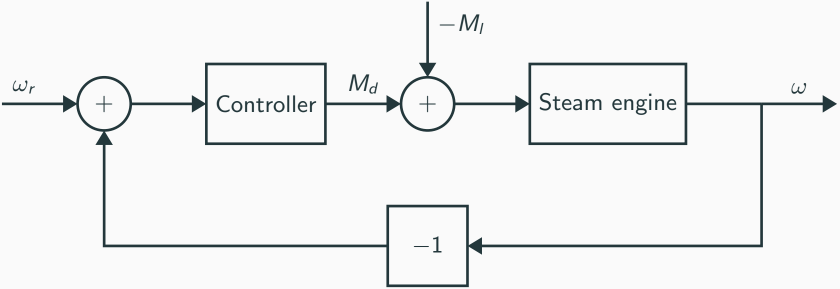
$$\omega(t) = \frac{M_d - M_l}{D}\left(1 - e^{-Dt/J}\right) = \omega_s\left(1 - e^{-Dt/J}\right)$$

Time constant:

$$T = \frac{J}{D}$$



# P Control of the Steam Engine




Proportional control:

$$M_d = K(\omega_r - \omega)$$

# P Control of the Steam Engine

Dynamics with P-controller:

$$J\dot{\omega} + D\omega = \overbrace{K(\omega_r - \omega)}^{M_d} - M_l$$

or

$$J\dot{\omega} + (D + K)\omega = K\omega_r - M_l$$

In stationarity ($\dot{\omega} = 0$):

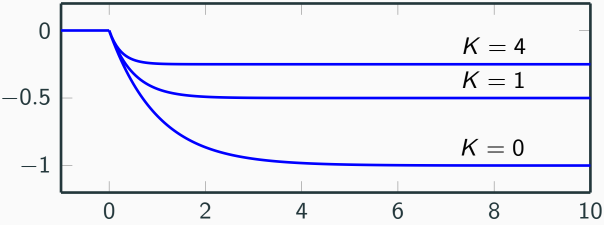
$$\omega_s = \frac{K}{D + K}\omega_r - \frac{1}{D + K}M_l$$

Step response ($\omega(0) = 0$):

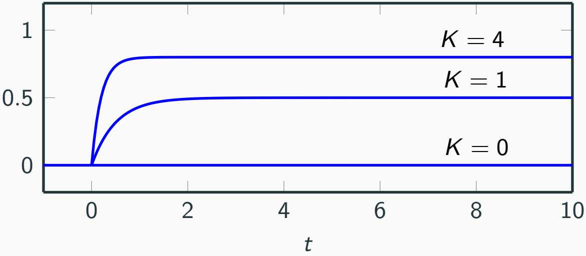
$$\omega(t) = \frac{K\omega_r - M_b}{D + K}\left(1 - e^{-(D+K)t/J}\right) = \omega_s\left(1 - e^{-(D+K)t/J}\right)$$

# P Control of the Steam Engine

Angular speed $\omega(t)$ ($\omega_r = 0$ and $M_l = \theta(t)$):




Driving torque $M_d(t)$ ($\omega_r = 0$ and $M_l = \theta(t)$):



# PI Control of the Steam Engine

Introduce a PI-controller to get rid of the stationary error:

$$M_d = K(\omega_r - \omega) + \frac{K}{T_i}\int_0^t (\omega_r - \omega)\mathrm{d}\tau$$

Dynamics:

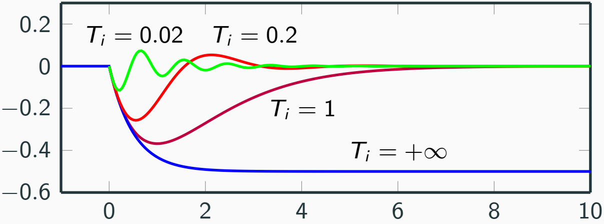
$$J\dot{\omega} + D\omega = K(\omega_r - \omega) + \frac{K}{T_i}\int_0^t (\omega_r - \omega)\mathrm{d}\tau - M_l$$
$$J\ddot{\omega} + D\dot{\omega} = K(\dot{\omega}_r - \dot{\omega}) + \frac{K}{T_i}(\omega_r - \omega) - \dot{M}_l$$

At stationarity ($\dot{\omega}_r = 0$, $\dot{M}_l = 0$):

$$\omega_s = \omega_r$$

# PI Control of the Steam Engine

Angular speed $\omega(t)$ ($\omega_r = 0$ and $M_l = \theta(t)$):

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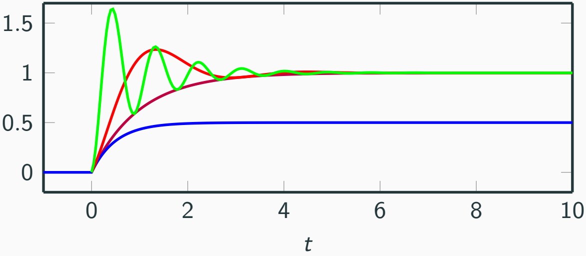


# PI Control of the Steam Engine

The Laplace transformation of the dynamics

$$J\ddot{\omega} + D\dot{\omega} = K(\dot{\omega}_r - \dot{\omega}) + \frac{K}{T_i}(\omega_r - \omega) - \dot{M}_l$$

is

$$s^2 J\omega + sD\omega = K(s\omega_r - s\omega) + \frac{K}{T_i}(\omega_r - \omega) - sM_l$$

The characteristic equation (the equation to determine the poles) is:

$$s^2 + \frac{D + K}{J}s + \frac{K}{J\,T_i} = 0$$

By choosing $K$ and $T_i$, we can place the poles of the closed loop dynamics arbitrarily.

# Stability

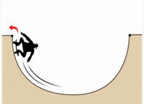
# Stability - Definitions

A system on state space form

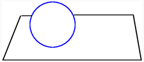
$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

is

**Asymptotically stable** if $x(t) \to 0$ when $t \to +\infty$ for all initial states $x(0)$ when $u(t) = 0$.



**Stable** if $x(t)$ is bounded for all $t$ and all initial states $x(0)$ when $u(t) = 0$.



**Unstable** if $x(t)$ grows unbounded for an initial state $x(0)$ when $u(t) = 0$.

biking on convex cobble-stone roads wintertime

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A system on state space form

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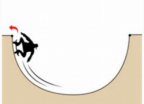

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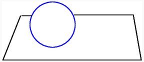

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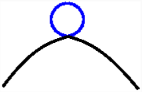


# Stability - Scalar Case

For the scalar case

$$\dot{x}(t) = ax(t)$$
$$x(0) = x_0$$

the solution is:

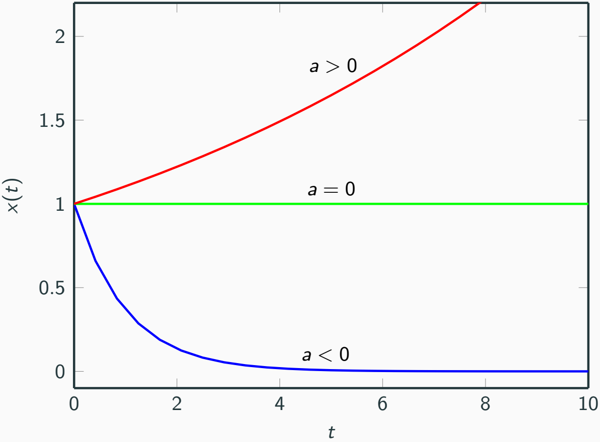
$$x(t) = e^{at} \cdot x_0$$

Hence

| | |
|---|---|
| $a < 0$ | Asymptotically stable |
| $a = 0$ | (marginally) Stable |
| $a > 0$ | Unstable |

# Stability - Scalar Case

$$\dot{x}(t) = ax(t), \quad x(0) = 1$$



# Stability - Diagonal Case

$$\dot{x}(t) = \overbrace{\begin{bmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_n \end{bmatrix}}^{A} x(t)$$

$$x(0) = x_0$$

Every state variable corresponds to the scalar case:

$$\dot{x}_i(t) = a_i x_i(t)$$

In fact, the $a_i$'s are **eigenvalues of $A$**. The system is

**Asymptotically stable** if all the eigenvalues of $A$ have negative real part

**Unstable** if at least one of the eigenvalues of $A$ has a positive real part

**(marginally) Stable** if all the eigenvalues of $A$ have either negative or zero real part

# Example diagonal case

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} u$$

$$y = \begin{bmatrix} 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

# Example diagonal case

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} u$$

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Look at each state individually (decoupled, i.e., does only depend on itself and input signal)

$$sX_1 - x_1(0) = -1X_1 + 4U$$
$$sX_2 - x_2(0) = +2X_2 + 5U$$
$$sX_3 - x_3(0) = -3X_3 + 6U$$

# Example diagonal case

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} u$$

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Look at each state individually (decoupled, i.e., does only depend on itself and input signal)

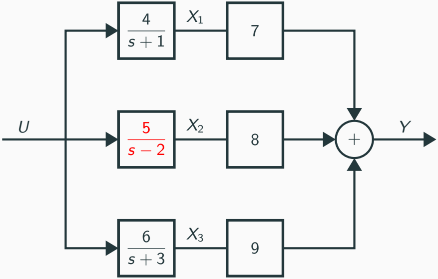
$$\begin{aligned}
sX_1 - x_1(0) &= -1X_1 + 4U \Longrightarrow & X_1 &= \frac{4}{s+1}\,U & &\underbrace{+\frac{4}{s+1}\,x_1(0)} \\
sX_2 - x_2(0) &= +2X_2 + 5U \Longrightarrow & X_2 &= \frac{5}{s-2}\,U & &+\frac{5}{s-2}\,x_2(0) \\
sX_3 - x_3(0) &= -3X_3 + 6U \Longrightarrow & X_3 &= \frac{6}{s+3}\,U & &\underbrace{+\frac{6}{s+3}\,x_3(0)}_{\textit{not in block diagram}}
\end{aligned}$$

...

# Example diagonal case

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} u$$

$$y = \begin{bmatrix} 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$




Stability is related to the whole system.

It is enough that one eigenvalue is in the RHP for the system to be unstable, **even if there would be no coupling to y!**

# Stability - General Case

For a general $A$-matrix, i.e., not necessarily a diagonal one, the stability rule still holds with one exception, namely that the eigenvalues having zero real part do not always guarantee stability, unless the purely imaginary eigenvalues are unique

Example

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

has a **double eigenvalue** at $\lambda = 0$.

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Example

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

has a **double eigenvalue** at $\lambda = 0$.

The differential equation $\dot{x} = Ax$ has the solution

$$x(t) = e^{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} t} x(0) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_1(0) + x_2(0) \cdot t \\ x_2(0) \end{bmatrix}$$

which **grows unbounded** for any $x_2(0) \neq 0$;

# Stability - Transfer Function

Recall from Lecture 2 that the eigenvalues of the $A$ matrix are poles to the transfer function. Hence, if all the poles have negative real part the system is stable.

A second order polynomial

$$s^2 + a_1 s + a_2$$

has its roots in the left half plane if and only if $a_1 > 0$ and $a_2 > 0$.

A third order polynomial

$$s^3 + a_1 s^2 + a_2 s + a_3$$

has its roots in the left half plane if $a_1 > 0$, $a_2 > 0$, $a_3 > 0$ and

$$a_1 a_2 > a_3$$

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$$a_1 a_2 > a_3$$

Stability critera wrt coeffients can be derived with **Routh-Hurwitz criterion**.

# Stability - Example

**Example**
Determine if the systems below are asymptotically stable or not

a)

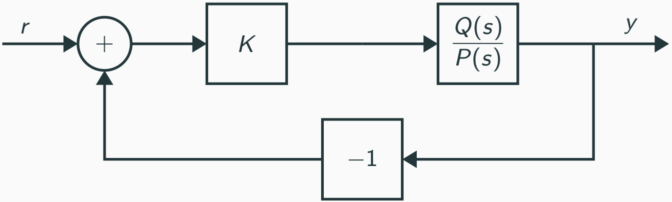
$$G(s) = \frac{1}{(s^2 + s + 1)(s + 1)}$$

b)

$$\dot{x} = \begin{bmatrix} -2 & 2 \\ 0 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 2 & -1 \end{bmatrix} x + 2u$$

# Root Locus

Idea: Study graphically how the poles move with the change of a parameter




$$Y(s) = \frac{KQ(s)}{P(s) + KQ(s)} R(s)$$

Characteristic equation:

$$P(s) + KQ(s) = 0$$

# Root Locus

Characteristic equation:

$$P(s) + KQ(s) = 0$$

For $K = 0$ the characteristic equation becomes:

$$P(s) = 0$$

When $K \to \infty$, the characteristic equation becomes:

$$Q(s) = 0$$

i.e., the poles of the closed loop system will approach the zeros of the closed loop system.

If there are more poles than zeros, the remaining poles will approach infinity (in magnitude).

# Root Locus - Second Order System

Let

$$\frac{Q(s)}{P(s)} = \frac{1}{s(s+1)}$$

Characteristic equation of the closed loop:

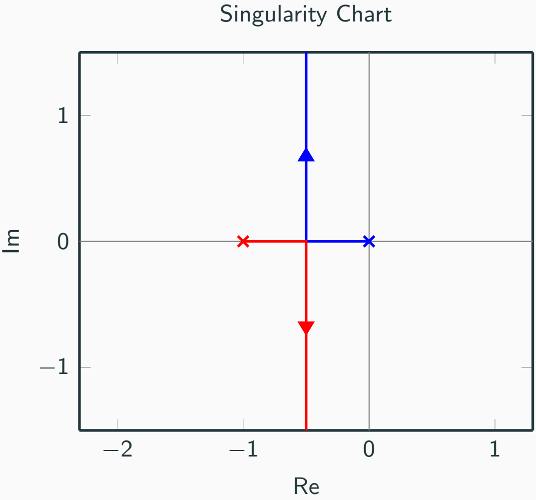
$$P(s) + KQ(s) = s(s+1) + K = 0$$

$$s = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - K}$$

When $K = 0$, poles in $0$, $-1$.

When $K > 1/4$, complex pair of poles with real part $-1/2$. The imaginary parts go towards $\pm\infty$ when $K \to \infty$.

# Root Locus - Second Order System



# Root Locus - Third Order System

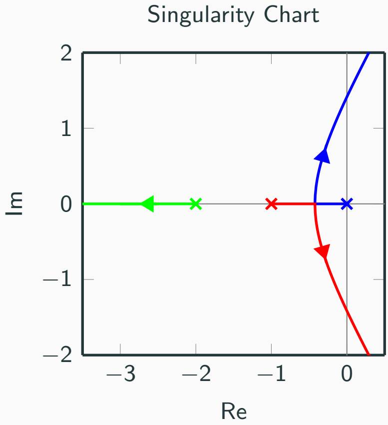
Let

$$\frac{Q(s)}{P(s)} = \frac{1}{s(s+1)(s+2)}$$

Characteristic equation of the closed loop:

$$P(s) + KQ(s) = s(s+1)(s+2) + K = s^3 + 3s^2 + 2s + K = 0$$

# Root Locus - Third Order System



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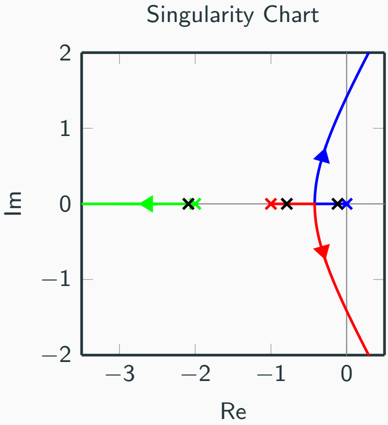


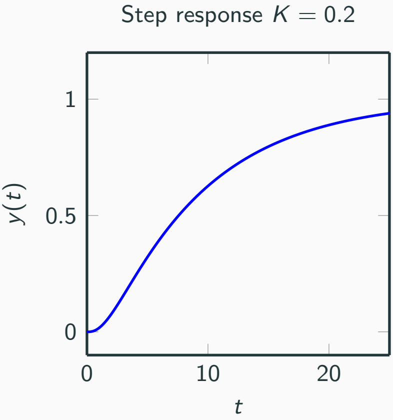

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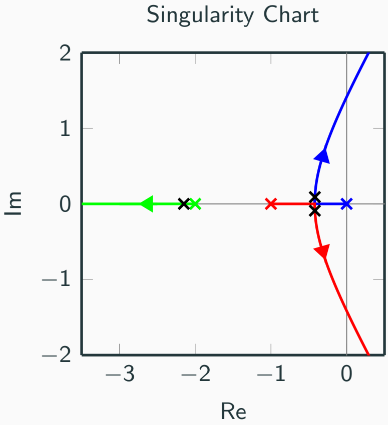


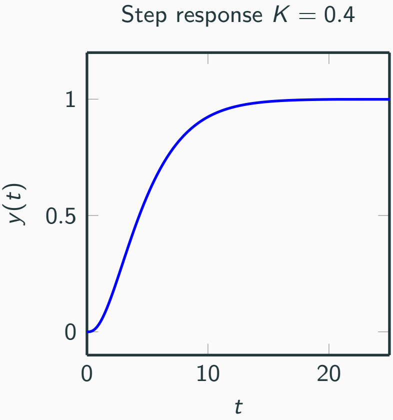

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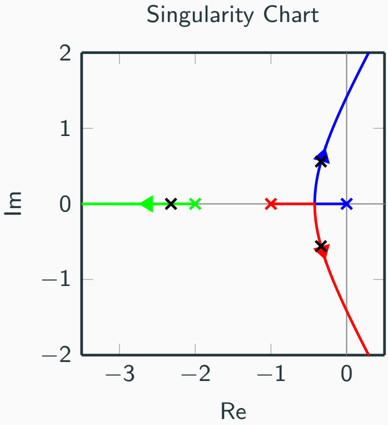


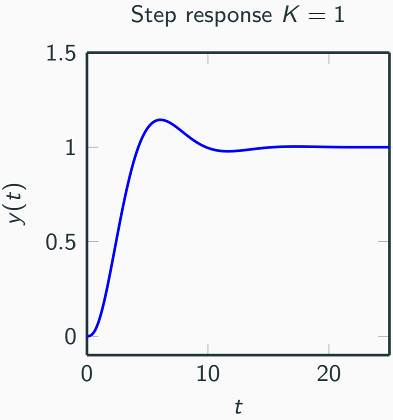

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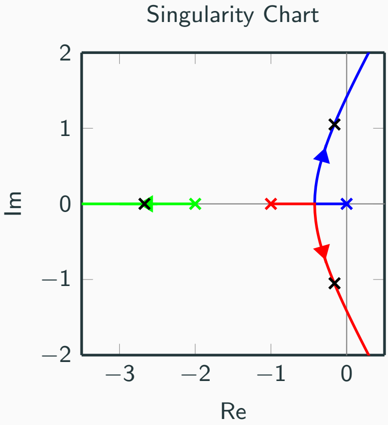


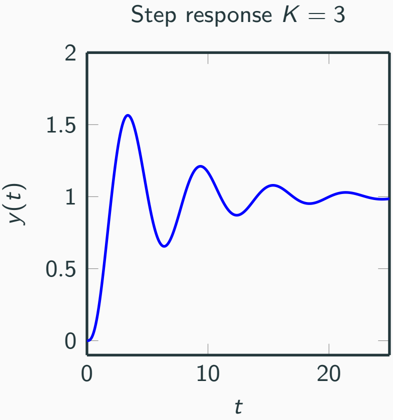

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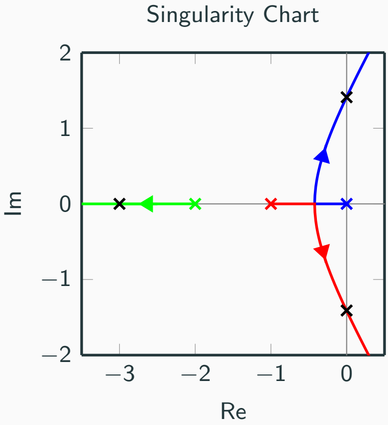


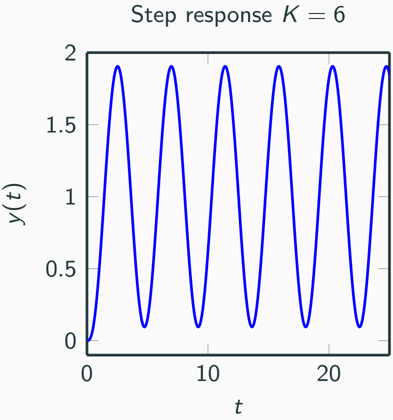

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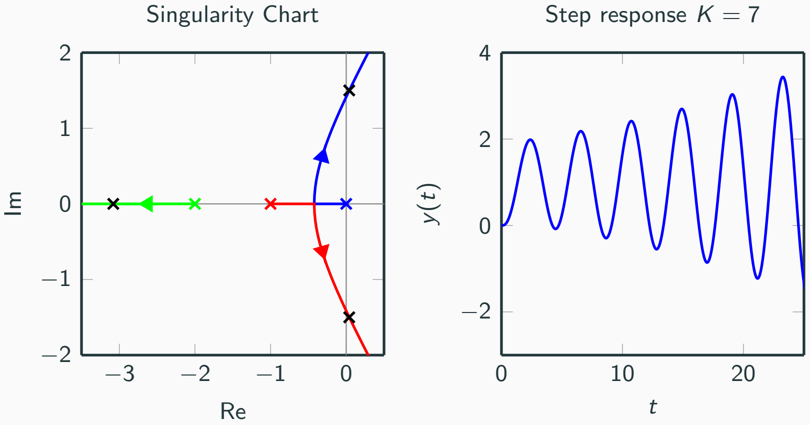
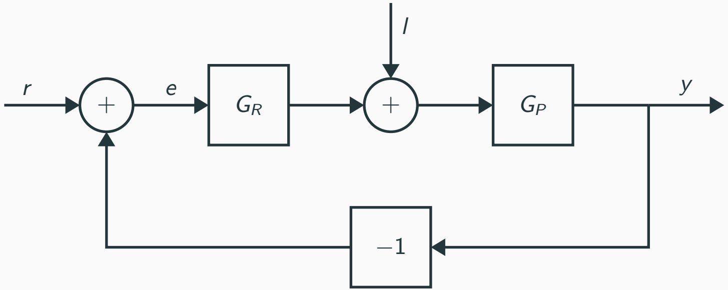


# Stationary Errors

# The Servo Problem and The Regulator Problem

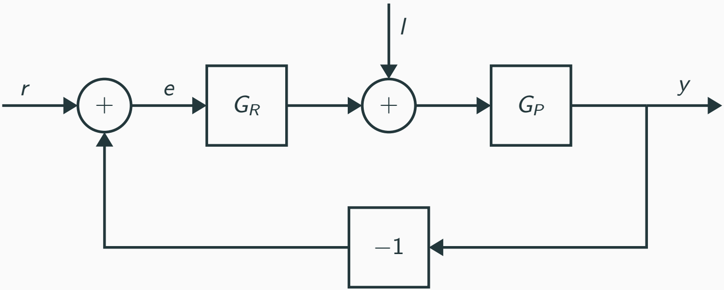



$$Y = \frac{G_R G_P}{1 + G_R G_P} R + \frac{G_P}{1 + G_R G_P} L$$

**The Servo Problem** Set point tracking of **r**, ($l = 0$).

**The Regulator Problem** Effect of load disturbances **l**, ($r = 0$).

# The Servo Problem and The Regulator Problem

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**The Servo Problem** Set point tracking of **r**, ($l = 0$).

**The Regulator Problem** Effect of load disturbances **l**, ($r = 0$).

Use **superposition property** of linear systems to consider them **separately**.

# Stationary Errors - The Servo Problem

$$E(s) = R(s) - Y(s) = \frac{1}{1 + \underbrace{G_R(s)G_P(s)}_{G_0(s)}} R(s)$$

We can use the final value theorem to determine the error

$$e_\infty = \lim_{t \to +\infty} e(t) = \lim_{s \to 0} sE(s)$$

but only if $sE(s)$ has it poles strictly in the left half plane.

# Stationary Errors - The Servo Problem - Example

Let the process and controller be:

$$G_P = \frac{1}{s(1+sT)} \quad G_R = K$$

Open-loop transfer function:

$$G_0 = G_R G_P = \frac{K}{s(s+sT)}$$

The control error is given by:

$$E(s) = \frac{1}{1+G_0(s)} R(s) = \frac{s(1+sT)}{s(1+sT)+K} R(s)$$

# Stationary Errors - The Servo Problem - Example

The control error is given by:

$$E(s) = \frac{1}{1 + G_0(s)} R(s) = \frac{s(1 + sT)}{s(1 + sT) + K} R(s)$$

Let $r(t)$ be a step, i.e.,

$$r(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases} \quad R(s) = \frac{1}{s}$$

Then (given that $T$ and $K$ are positive)

$$e_\infty = \lim_{t \to +\infty} e(t) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} s \cdot \frac{s(1 + sT)}{s(1 + sT) + K} \cdot \frac{1}{s} = 0$$

# Stationary Errors - The Servo Problem - Example

The control error is given by:

$$E(s) = \frac{1}{1 + G_0(s)} R(s) = \frac{s(1 + sT)}{s(1 + sT) + K} R(s)$$

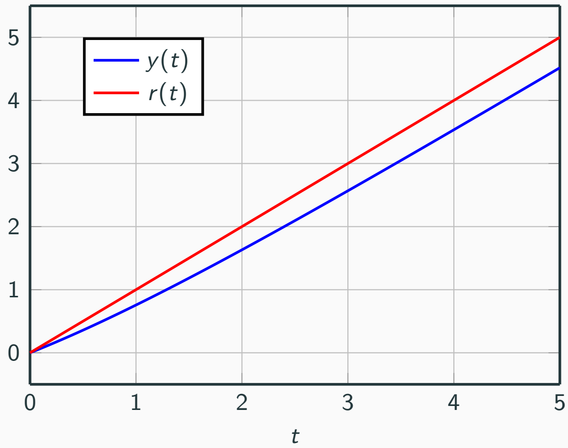
Let $r(t)$ be a ramp, i.e.,

$$r(t) = \begin{cases} t & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases} \quad R(s) = \frac{1}{s^2}$$

Then (given that $T$ and $K$ are positive)

$$e_\infty = \lim_{t \to +\infty} e(t) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} s \cdot \frac{s(1 + sT)}{s(1 + sT) + K} \cdot \frac{1}{s^2} = \frac{1}{K}$$

# Stationary Errors - The Servo Problem - Example




Question to the audience: What value of $K$ is used?

# Stationary Errors - The Servo Problem - General Case

Open loop transfer function:

$$G_0(s) = \frac{K}{s^n} \cdot \frac{1 + b_1 s + b_2 s^2 + \dots}{1 + a_1 s + a_2 s^2 + \dots} e^{-sL} = \frac{KB(s)}{s^n A(s)} e^{-sL}$$

Set point ($m$ non-negative integer):

$$r(t) = \begin{cases} \frac{t^m}{m!} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases} \qquad R(s) = \frac{1}{s^{m+1}}$$

Error (given that the limit exists):

$$e_\infty = \lim_{t \to +\infty} e(t) = \lim_{s \to 0} s \cdot \frac{s^n A(s)}{s^n A(s) + KB(s) e^{-sL}} \cdot \frac{1}{s^{m+1}} = \lim_{s \to 0} \frac{1}{s^n + K} s^{n-m}$$

The stationary error is determined by the low-frequency properties of the transfer function and the set point.

# Stationary Errors - The Servo Problem - General Case

$$G_0(s) = \frac{K}{s^n} \cdot \frac{1 + b_1 s + b_2 s^2 + \dots}{1 + a_1 s + a_2 s^2 + \dots} e^{-sL} = \frac{KB(s)}{s^n A(s)} e^{-sL}$$

$$r(t) = \begin{cases} \frac{t^m}{m!} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

The relation between $m$ and $n$ gives the following errors:

$$\begin{array}{cl} n > m & e_\infty = 0 \\ n = m = 0 & e_\infty = \frac{1}{1+K} \\ n = m \geq 1 & e_\infty = \frac{1}{K} \\ n < m & \text{Limit does not exist.} \end{array}$$

# Stationary Errors - The Regulator Problem

The transfer function between a load disturbance $l(t)$ and measurement signal $y(t)$:

$$Y(s) = \frac{G_P(s)}{1 + G_R(s)G_P(s)} L(s)$$

Since $r = 0$, we can study the measurement signal instead of the error:

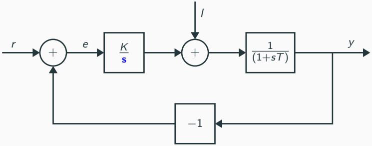
$$y_\infty = \lim_{t \to +\infty} y(t) = \lim_{s \to 0} sY(s)$$

Again, we have to ensure that the limit exists.

# Stationary Errors - The Regulator Problem - Example

Let the controller and process be:

$$G_P = \frac{1}{1+sT}, \qquad G_R = \frac{K}{\mathbf{s}}$$




Let the load disturbance $l(t)$ be a step:

$$l(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

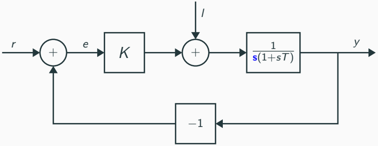
The final theorem yields:

$$y_\infty = \lim_{t \to +\infty} y(t) = \lim_{s \to 0} s \cdot \frac{\mathbf{s}}{\mathbf{s}(1+sT)+K} \cdot \frac{1}{s} = 0$$

# Stationary Errors - The Regulator Problem - Example

Let the controller and process instead be:

$$G_R = K, \qquad G_P = \frac{1}{\mathbf{s}(1+sT)}$$




Notice that $G_0 = G_P G_R$ is the same as in the previous slide. Let the load disturbance $l(t)$ be a step:

$$l(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

The final theorem yields:

$$y_\infty = \lim_{t \to +\infty} y(t) = \lim_{s \to 0} s \frac{1}{\mathbf{s}(1+sT)+K} \cdot \frac{1}{s} = \frac{1}{K}$$

# Stationary Errors - The Regulator Problem - Example

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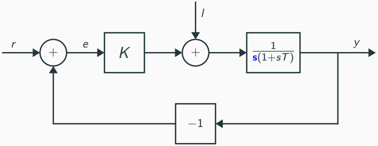


$$G_R = K, \qquad G_P = \frac{1}{\mathbf{s}(1+sT)}$$

Notice that $G_0 = G_P G_R$ is the same as in the previous slide. Let the load disturbance $l(t)$ be a step:

$$l(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

The final theorem yields:

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In the regulator problem, **the placement of integrators matters (i.e., if integrators are in controller or in plant)**.

# Stationary Errors - The Regulator Problem - General Case

Let

$$G_P(s) = \frac{K_P B_P(s)}{s^p A_p(s)} e^{-sL} \quad G_R(s) = \frac{K B_R(s)}{s^r A_R(s)}$$

where $A_P(0) = B_P(0) = A_R(0) = B_R(0) = 1$. Moreover, let the load disturbances be given by

$$L(s) = \frac{1}{s^{m+1}}$$

Then

$$y_\infty = \lim_{s \to 0} \frac{K_P}{s^{r+p} + K K_P} s^{r-m}$$

The stationary becomes (given that the limits exists):

$$\begin{array}{cl} r > m & y_\infty = 0 \\ r = m = 0,\ p = 0 & y_\infty = \frac{K_P}{1 + K K_P} \\ r = m = 0,\ p \geq 0 & y_\infty = \frac{1}{K} \\ r = m \geq 1 & y_\infty = \frac{1}{K} \\ r < m & \text{The limit does not exist.} \end{array}$$

# Stationary Errors - Example

**Example**
The transfer function of a process is

$$G_p(s) = \frac{1}{s+1}.$$

The process is controlled with a PI-regulator,

$$G_r(s) = 1 + \frac{2}{s}.$$

The closed loop system is able to follow step changes in the reference value without any stationary error, but when the reference is a ramp-signal, $r(t) = ct$, a stationary error emerges. Determine the magnitude of this stationary error.

# Content

This lecture

1. Feedback – The Steam Engine

2. Stability

3. Stationary Errors

Next lecture

- Nyquist criterion
- Stability margins
- Sensitivity function