Lec 5: Frequency Domain Stability Analysis

The Nyquist Criterion. Stability Margins. Sensitivity

November 20, 2018

Lund University, Department of Automatic Control

Content

Stability is Important!

Stability Margins are also Important!

X29

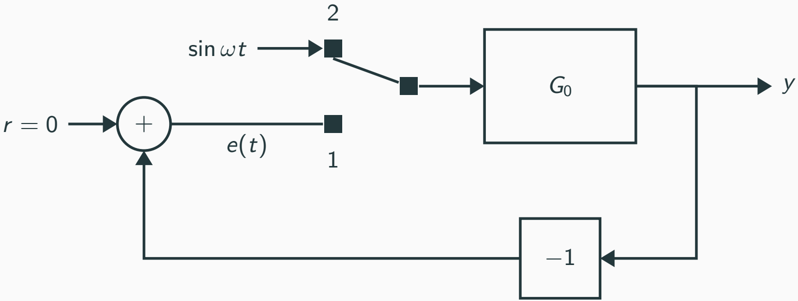
Harry Nyquist (1889-1976)

Nilsby, Sweden $\rightarrow$ North Dakota $\rightarrow$ Yale $\rightarrow$ Bell Labs

- Nyquist's stability criterion
- The Nyquist frequency
- Johnson-Nyquist noise

Nyquist's Criterion

Nyquist's Criterion — A Motivation



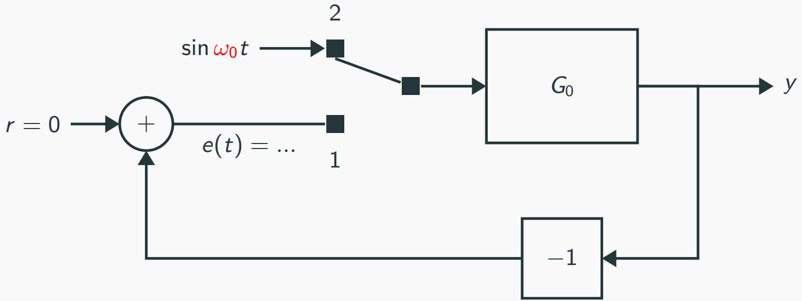
With switch in position 2, after transients (G_0 stable):

$$e(t) = -|G_0(i\omega)| \sin(\omega t + \arg G_0(i\omega)) \\ = |G_0(i\omega)| \sin(\omega t + \arg G_0(i\omega) + \pi)$$

Find ω_0 such that $\arg G_0(i\omega_0) = -\pi$.

Also assume $|G_0(i\omega_0)| = 1$

Nyquist's Criterion — A Motivation



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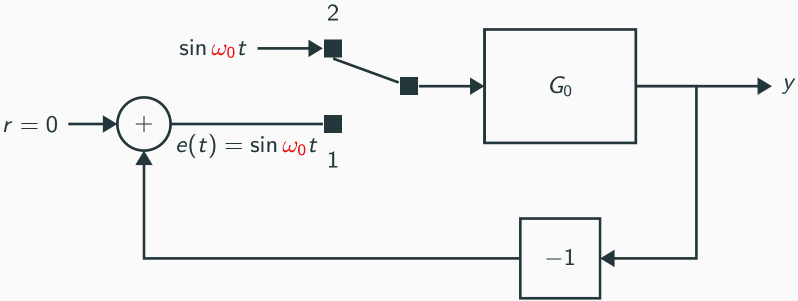

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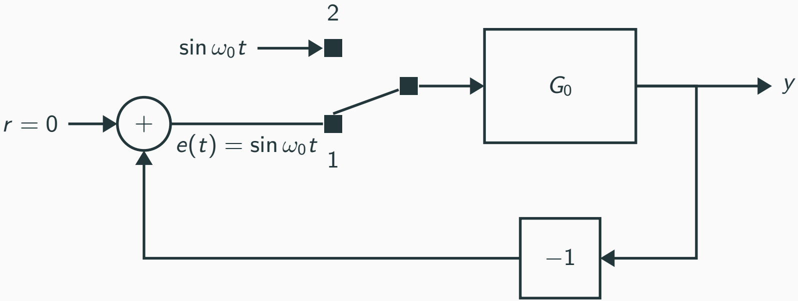
Nyquist's Criterion — A Motivation



Oscillation will continue in closed loop. We have a marginally stable system.

Seems likely that

- $|G_0(i\omega_0)| < 1 \Rightarrow$ Oscillation damped out (Asymptotic stability)
- $|G_0(i\omega_0)| > 1 \Rightarrow$ Oscillation increases (Instability)

Bode and Nyquist diagrams

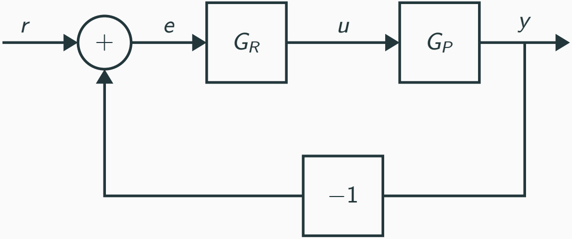
We **most often** plot Bode and Nyquist diagrams for "the open-loop system" G_O (aka *loop gain* L)

$$L = G_O = G_R G_p$$

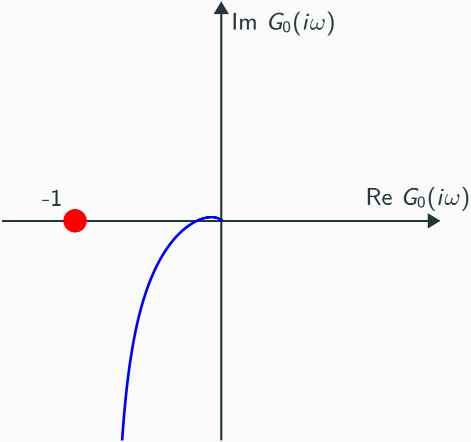
and from this predict how the closed-loop system

$$\frac{G_R G_p}{1 + G_R G_p}$$

will behave.



Nyquist's Criterion

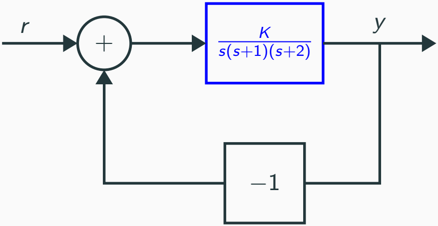


Nyquist's Criterion (simplified version):

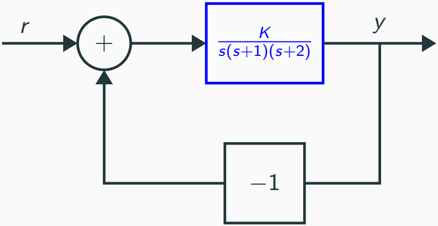
Assume $G_0(s)$ is stable.

Then the closed loop system (simple negative feedback) is stable if the point -1 lies to the left of $G(i\omega)$ as ω goes from 0 to ∞.

Example



Example

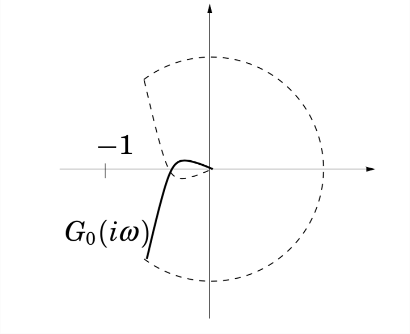


Loop gain (Open system)

$$
\begin{aligned}
G_0(i\omega) &= \frac{K}{i\omega(1+i\omega)(2+i\omega)} \\
&= \frac{-Ki(1-i\omega)(2-i\omega)}{\omega(1+\omega^2)(4+\omega^2)} = \frac{-Ki(2-\omega^2-3i\omega)}{\omega(1+\omega^2)(4+\omega^2)} \\
&= \frac{-3K}{(1+\omega^2)(4+\omega^2)} + i\frac{K(\omega^2-2)}{\omega(1+\omega^2)(4+\omega^2)}
\end{aligned}
$$

$$\lim_{R\to\infty} G_0(Re^{i\phi}) = 0 \qquad \lim_{r\to 0} G_0(re^{i\phi}) = \frac{K}{2r}e^{-i\phi}$$

Stability for closed-loop system



Crossing with negative real axis:

$$\text{Phase} = \text{-180 deg} \Longrightarrow \mathrm{Im}\left\{G_O(i\omega_0)\right\} = 0 \Longrightarrow \underline{\omega_0 = \sqrt{2}}$$

$$G_0(i\sqrt{2}) = -\frac{3K}{3 \cdot 6} = -\frac{K}{6}$$

Stable if $K < 6$. Two poles in right half-plane if $K > 6$.

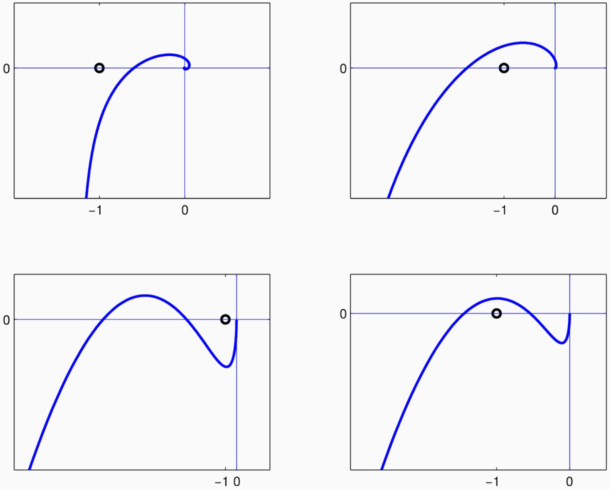
Nyquist's criterion — Some comments

- Gives insight
- Easy to use, only requires frequency response
- Slightly complex to prove
- Version of Nyquist Criterion also works if $G_0(s)$ is unstable.

Quiz

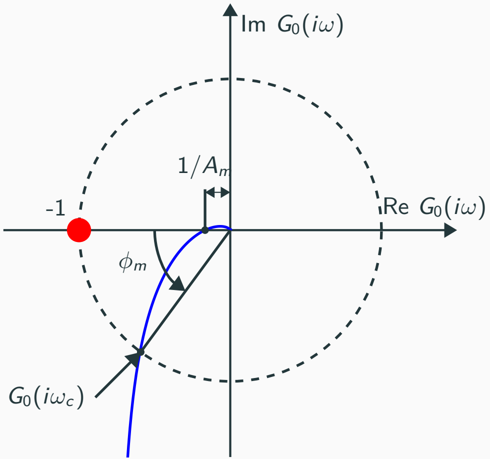
Nyquist curves of four (open-loop stable) systems.

Which systems are stable in closed loop (simple negative feedback)?



Stability Margins

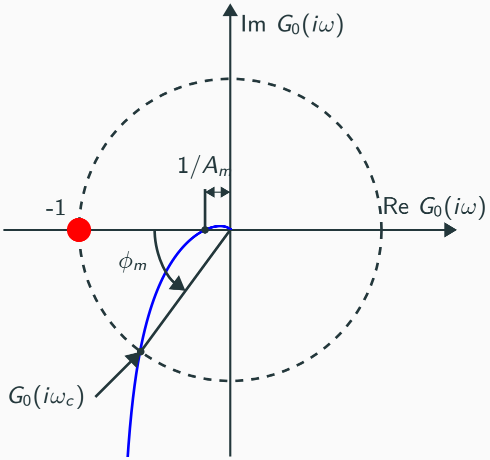
Stability Margin



Amplitude margin: "Gain increase without instability"

Phase margin: "Phase decrease without instability"

Stability Margin



Important with sufficient stability margins for good control performance

Rule of thumb: $A_m > 2$, $\phi_m > 45^\circ$

Delay Margin

Augment open-loop transfer function $G_0(s)$ with a delay L:

$$G_0^{new}(s) = e^{-sL} G_0(s)$$

We have

$$|G_0^{new}(i\omega)| = |G_0(i\omega)|$$

$$\arg G_0^{new}(i\omega) = \arg G_0(i\omega) - \omega L$$

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Same cross-over frequency ω_c as G_0, so new phase margin

$$\varphi_m^{new} = \varphi_m - \omega_c L$$

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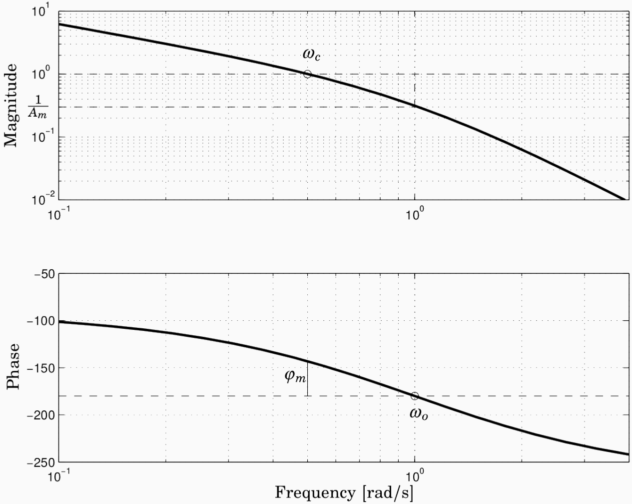
Same cross-over frequency ω_c as G_0, so new phase margin

$$\varphi_m^{new} = \varphi_m - \omega_c L$$

For stability the delay L must be smaller than

$$L_m = \frac{\varphi_m}{\omega_c}$$

Amplitude & Gain Margins in Bode Plots



ω_c is called the cross-over frequency.

Sensitivity Function

The Sensitivity Function

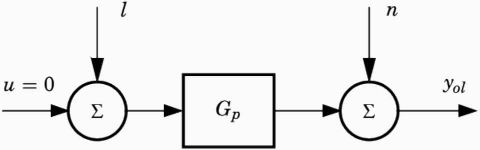
The closed-loop transfer function

$$S(s) = \frac{1}{1 + G_R(s)G_P(s)}$$

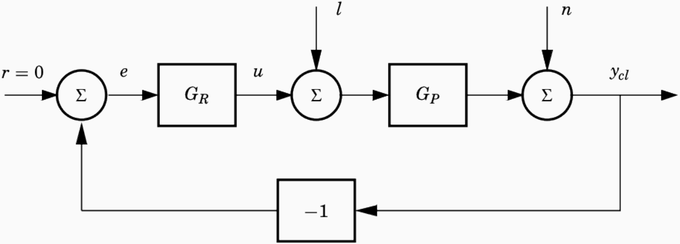
is called the **sensitivity function**.

Gives much information about closed-loop control performance.

Interpretation of Sensitivity Function (1/3)

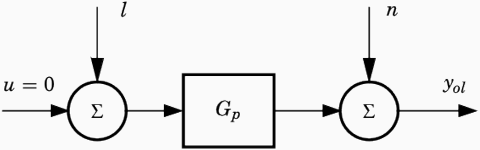


$$Y_{ol}(s) = \dots\ L(s) + \dots\ N(s)$$

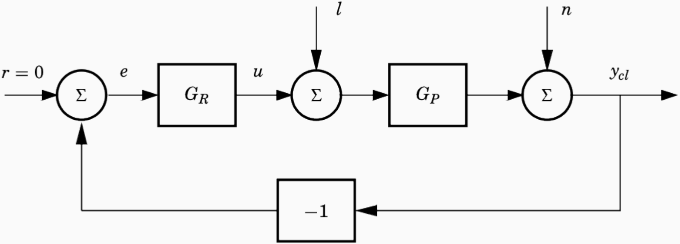


$$Y_{cl}(s) = \dots\ L(s) + \dots\ N(s)$$

Interpretation of Sensitivity Function (1/3)



$$Y_{cl}(s) = G_P(s)L(s) + 1 \cdot N(s)$$



$$Y_{cl}(s) = \frac{G_P(s)}{1 + G_R(s)G_P(s)}L(s) + \frac{1}{1 + G_R(s)G_P(s)}N(s)$$

Interpretation of Sensitivity Function (1/3)

$$Y_{ol}(s) = G_P(s)L(s) + 1 \cdot N(s)$$

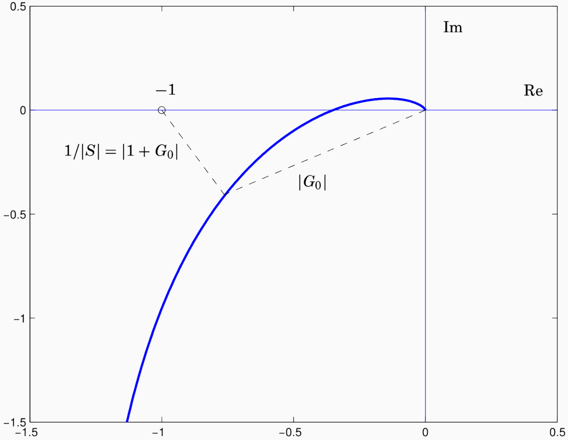
$$Y_{cl}(s) = \frac{G_P(s)}{1 + G_R(s)G_P(s)} L(s) + \frac{1}{1 + G_R(s)G_P(s)} N(s)$$

The sensitivity function quantifies the effect of feedback.

$|S(i\omega)| < 1 \Rightarrow$ disturbances with frequency ω are reduced by controller
$|S(i\omega)| > 1 \Rightarrow$ disturbances with frequency ω are magnified by controller

Typically the controller will always increase disturbances at some frequencies. Preferably not at frequencies with much disturbances.

Interpretation of Sensitivity Function (2/3)



$1/|S(i\omega)|$ is the distance between the Nyquist curve and -1.

$M_s = \sup_\omega |S(i\omega)|$ can be used to quantify the stability margin.

Interpretation of Sensitivity Function (3/3)

The sensitivity function quantifies closed-loop sensitivity to modeling errors. Let G_P be our process model.

$$G_P^0 = G_P(1 + \Delta G)$$

G_P^0 is the actual process dynamics, ΔG is the relative modeling error .

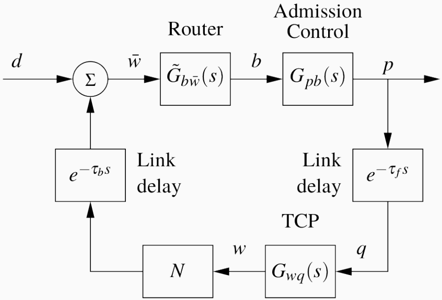
Can show that

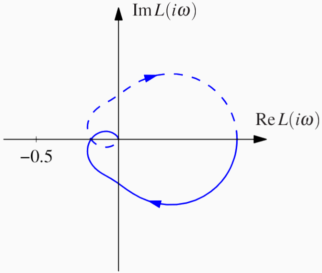
$$Y^0 = \left(1 + S^0 \Delta G\right) Y$$

S^0 is the sensitivity function of the *real* system.

$$\frac{Y^0 - Y}{Y} = S^0 \Delta G$$

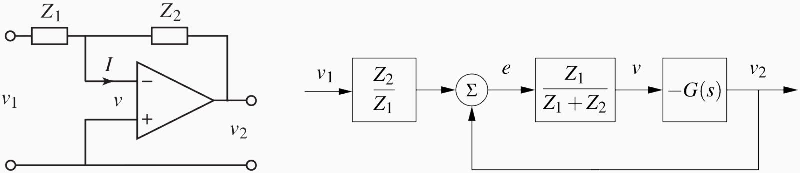
Example: Internet Congestion Control





See Example 9.5 in [Åström & Murray] for details.

Example: Operational Amplifier



(a) Amplifier circuit (b) Block diagram

Transfer function from v_1 to v_2;

$$G_{cl}(i\omega) = -\frac{Z_2}{Z_1}\frac{Z_1 G(i\omega)/(Z_1+Z_2)}{1+Z_1 G(i\omega)/(Z_1+Z_2)}$$

$\approx -Z_2/Z_1$ (If closed loop is stable, and ω within bandwidth)

What about stability? Just look at Nyquist curve of

$$G_o(s) = \frac{Z_1 G(s)}{Z_1+Z_2}$$

Don't need model of the op-amp, just measured transfer function!
(Power of Nyquist's Criterion)

Content

This lecture

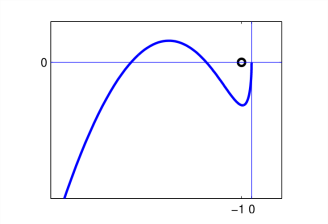
1. Nyquist's Criterion
2. Stability Margins
3. Sensitivity Function

Next lecture

- State feedback
- Controllability
- Integral action

Extra

- Nyquist criterion (general case)
- Non-intuitive stability case (from Quiz)



Based on Chapter 11 of

`http://www.cds.caltech.edu/∼murray/amwiki/`

Nyquist's stability theorem

Consider a closed loop system with the loop transfer function $G_o(s)$ that has P poles in the region enclosed by the Nyquist contour. Let N be the net number of clockwise encirclements of $-1+0i$ by $G_o(s)$ when s encircles the Nyquist contour C in the clockwise direction. The closed loop system then has $Z = N + P$ poles in the right half-plane.

Nyquist criterion and Cauchy's argument variation principle

Nyquist's stability theorem

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Note: We are considering an open loop transfer function, the loop gain

$$L = G_o$$

to conclude about the stability of the closed loop system

$$G_{cl} = \frac{L}{1+L} = \frac{G_o}{1+G_o}$$

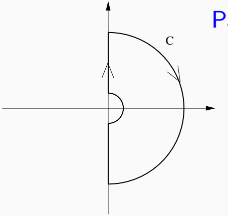
.

Cauchy's argument variation principle

How many zeros does a rational function $f(\cdot)$ have in a region C?

$$\frac{1}{2\pi}\Delta_{s\in C} \arg f(s) = P - N$$

To determine the number of roots in the **right half plane** we choose the closed curve C in the following way



Paths for C:

$s = i\omega,\ \omega \in (0, \infty)$

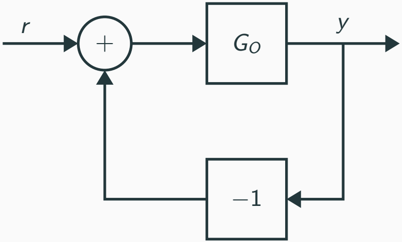
$s = Re^{i\theta},\ R \to \infty, \theta : \pi/2 \to -\pi/2$

$s = i\omega,\ \omega \in (-\infty, 0)$

$s = re^{i\theta},\ r \to 0, \theta : -\pi/2 \to \pi/2$

Small half-circle around the origin avoids singularities on the boundary, (e.g., in case of integrator $1/s$ in the loop gain)

Stability for feedback

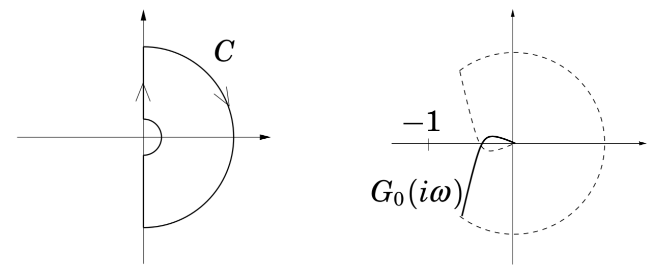


The closed-loop system $\dfrac{G_o}{1+G_o}$ is asymptotically stable if and only if all zeros to

$$1 + G_0(s)$$

are in the left half-plane.

Cauchy's argument variation principle for feedback



$$N = \#\text{ zeros for } 1 + G_0(s) \text{ inside curve } C$$

$$P = \#\text{ poles for } 1 + G_0(s) \text{ inside curve } C$$

$$= \#\text{ poles for } G_0(s) \text{ inside curve } C$$

Argument variation principle gives

$$P - N = \#\text{ rev. around origin for } 1 + G_0(s),\ s \in C$$

$$= \#\text{ rev. around } -1 + 0i \text{ for } G_0(i\omega),\ \omega \in \mathbf{R}$$

Nyquist criterion

(Simplified):
If $G_o(s)$ is stable ($P = 0$), then the closed-loop system

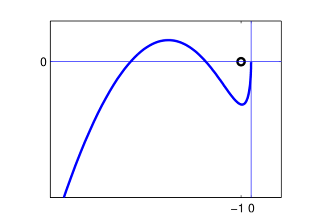
$$\frac{G_o}{1 + G_o}$$

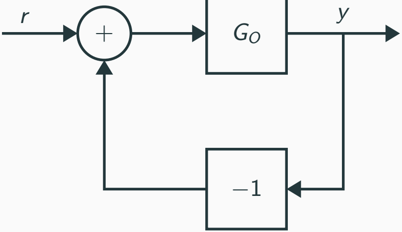
is stable ($N = 0$) if and only if the Nyquist-curve $G_o(i\omega)$ does NOT encircle $-1 + 0i$.

(General):
The difference between the number of unstable poles in $G_o(s)$ and the number of unstable poles in $G_o/(1 + G_o)$ is equal to the number of turns of the Nyquist-curve around $-1 + 0i$. (Note: direction important if counted as positive or negative turns!)

When 'intuition' doesn't hold, we rely on mathematics

From the Quiz of Lecture 5 we asked for which loop gain systems under simple negative feedback which gave stable closed-loop systems.

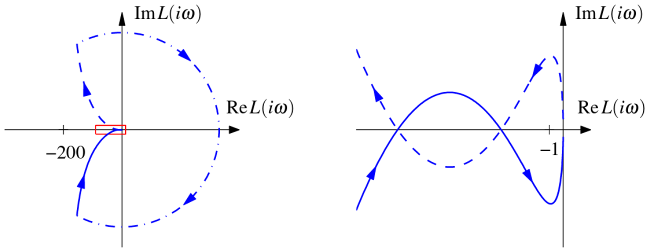




What happens with the system in the picture *(left)* when we close the loop and why is it still stable?

When 'intuition' doesn't hold, we rely on mathematics

Look at the loop gain $L = G_o(s) = 3 \cdot \frac{1}{s} \cdot \frac{1}{(s+1)^2} \cdot (s+6)^2$



(a) Nyquist plot (b) Enlargement around -1

The first intersection with negative real axis occurs at

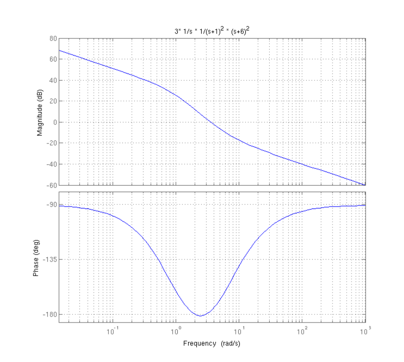
$$G_o(i\omega) = -12 \text{ for } \omega = 2,$$

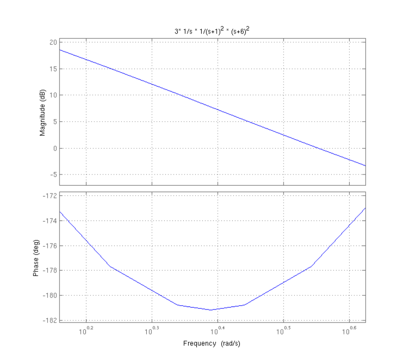
and the second at

$$G_o(i\omega) = -4.5 \text{ for } \omega = 3.$$

When 'intuition' doesn't hold, we rely on mathematics

Look at the loop gain $G_o(s) = 3 \cdot \frac{1}{s} \cdot \frac{1}{(s+1)^2} \cdot (s+6)^2$





Closed-loop system

$$G_{cl} = \frac{G_o(s)}{1+G_o(s)} = \frac{3 \cdot \dfrac{1}{s} \cdot \dfrac{1}{(s+1)^2} \cdot (s+6)^2}{1 + 3 \cdot \dfrac{1}{s} \cdot \dfrac{1}{(s+1)^2} \cdot (s+6)^2} = \frac{s^2+12s+36}{s^3+3s^2+13s+36}$$

Closed-loop system

$$G_{cl} = \frac{G_o(s)}{1 + G_o(s)} = \frac{3 \cdot \frac{1}{s} \cdot \frac{1}{(s+1)^2} \cdot (s+6)^2}{1 + 3 \cdot \frac{1}{s} \cdot \frac{1}{(s+1)^2} \cdot (s+6)^2} = \frac{s^2 + 12s + 36}{s^3 + 3s^2 + 13s + 36}$$

Stability can be tested with e.g., Routh-Hurwitz criterion

(i) $a_1 = 3 > 0$

$a_2 = 13 > 0$

$a_3 = 36 > 0$

(ii) $a_1 \cdot a_2 > a_3 \qquad (3 \cdot 13 = 39 > 36)$

and thus the closed-loop system is **asymptotically stable.**

```
» roots([1 3 13 36])
  ans =
      -0.0709 + 3.5482i
      -0.0709 - 3.5482i
      -2.83
```