Lec 6: State Feedback, Controllability, Integral Action

November 21, 2018

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Content

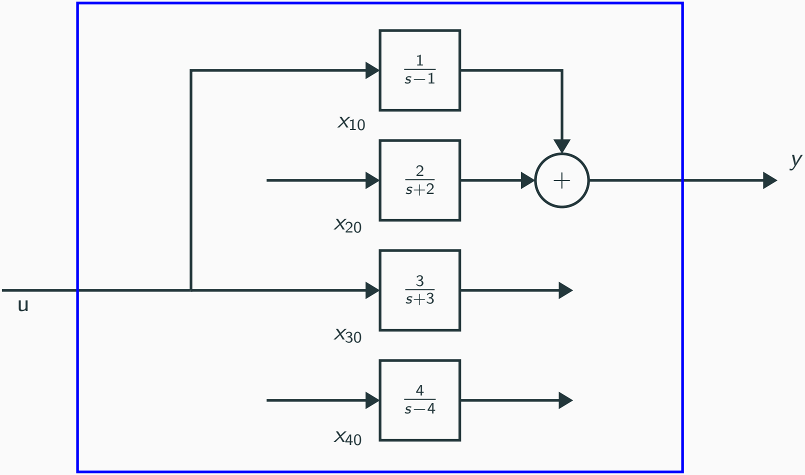
This lecture

1. Controllable form
2. State feedback control
3. Example
4. Controllability
5. Integral Action

Next lecture: Almost the same but about *observability*

Controllability and Observability

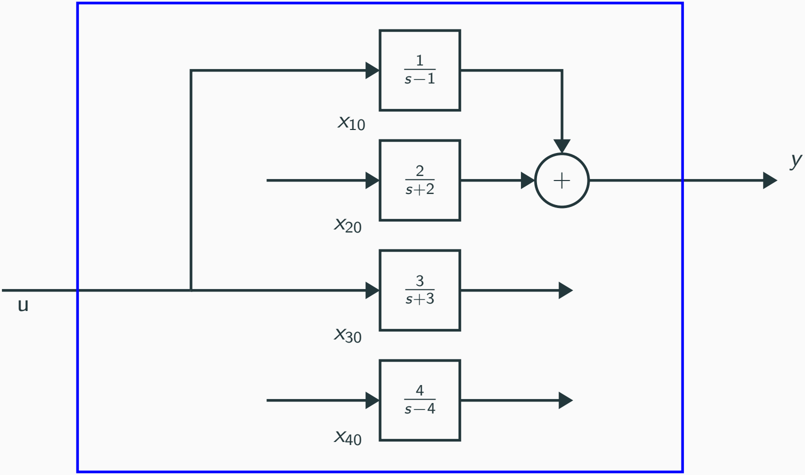
Example of Kalman decomposition



How does the system behave? From outside the blue box we only see the input u and output y but **a lot can happen inside!!**

Controllability and Observability

Example of Kalman decomposition

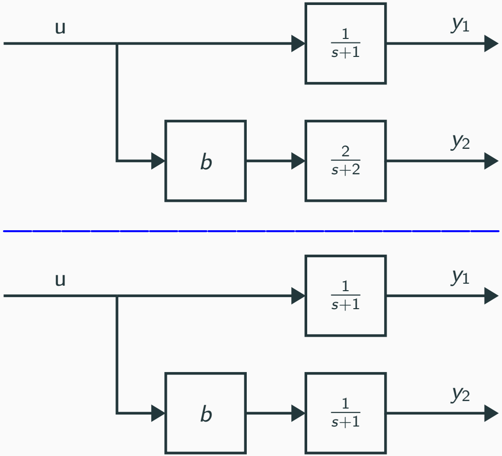


Introduce states and write the system in state-space form.

Controllability and Observability

How well can we control two subsystems at the same time?

- Q: Are there any differences for the two cases below and how does it depend on gain b?



Different Ways to Describe a Dynamical System

Differential equation

$$\ddot{y} + a_1\dot{y} + a_2 y = b_1\dot{u} + b_2 u$$

$$x_1 = y$$
$$x_2 = \dot{y}$$
$$\dots$$

$$\mathcal{L}\left(\frac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}\right)(s) = s^n F(s)$$

Collection of Formulae

State space

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Transfer function

$$Y(s) = G(s)U(s) = \frac{Q(s)}{P(s)}U(s)$$

$$G(s) = C(sI - A)^{-1}B + D$$

Recap of states and statespace realization

Physical systems are modeled by differential equations.

Example: Damped spring-mass system ($\dot{y} \leftrightarrow$ velocity, $y \leftrightarrow$ position)

$$m\ddot{y}(t) = -c\dot{y}(t) - ky(t) + F(t)$$

The state vector x is a collection of physical quantities required to predict the evolution of the system

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

x_i's could be positions, velocities, currents, voltages, queue lengths, number of virtual machines, temperatures, concentrations, etc.

Recap of states and statespace realization

The evolution of the state vector (for an LTI system), subject to an external signal u, can be described by

$$\dot{x} = Ax + Bu$$

where A is a matrix and B a column vector.

The measured signal of the system is given by

$$y = Cx \; (+ Du)$$

Statespace $\leftrightarrow$ Transfer function conversion

A system with state-space representation

$$\dot{x} = Ax + Bu$$
$$y = Cx \ (+Du)$$

has transfer function

$$G(s) = C(sI - A)^{-1}B + D$$

A transfer function model can have (infinitely) many different state-space realizations.

Standard forms are: **controllable canonical form, observable canonical form, diagonal form**, see Collection of Formulae.

Lecture 6: State feedback control

1. Controllable form
2. State feedback control
3. Example
4. Controllability
5. Integral Action

Controllable canonical form

The system with transfer function

$$G(s) = D + \frac{b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n}$$

has controllable canonical form

$$\frac{dz}{dt} = \begin{bmatrix} -a_1 & -a_2 & \dots & -a_{n-1} & -a_n \\ 1 & 0 & & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & & 1 & 0 \end{bmatrix} z + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & b_2 & \dots & b_n \end{bmatrix} z + Du$$

Observable canonical form

The system with transfer function

$$G(s) = D + \frac{b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n}$$

has observable canonical form

$$\frac{dz}{dt} = \begin{bmatrix} -a_1 & 1 & 0 & \dots & 0 \\ -a_2 & 0 & 1 & & 0 \\ \vdots & & & & \\ -a_{n-1} & 0 & 0 & & 1 \\ -a_n & 0 & 0 & & 0 \end{bmatrix} z + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ \\ b_n \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \end{bmatrix} z + Du$$

PID and state feedback controller

PID-controller:

$$u = K\left(e + \frac{1}{T_i}\int^t e(\tau)\mathrm{d}\tau + T_d\frac{\mathrm{d}e}{\mathrm{d}t}\right), \qquad e = r - y$$

State feedback controller

$$u = l_{\text{ref}}\, r + l_1(x_{1,ref} - x_1) + l_2(x_{2,ref} - x_2) - \cdots + l_n(x_{n,ref} - x_n)$$

PID and state feedback controller

PID-controller:

$$u = K\left(e + \frac{1}{T_i}\int^t e(\tau)\mathrm{d}\tau + T_d\frac{\mathrm{d}e}{\mathrm{d}t}\right), \qquad e = r - y$$

State feedback controller

$$\begin{aligned} u &= l_{\text{ref}}\, r + l_1(x_{1,ref} - x_1) + l_2(x_{2,ref} - x_2) - \cdots + l_n(x_{n,ref} - x_n) \\ &= \\ &= l_{\text{ref}}\, r - l_1 x_1 - l_2 x_2 - \cdots - l_n x_n \end{aligned}$$

if $x_{1,ref} = x_{2,ref} = ... = x_{n,ref} = 0$

We will also add integral part to the state feedback later on.

Process

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

$$Y(s) = C(sI - A)^{-1}BU(s)$$

The characteristic polynomial is $\det(sI - A)$

Controller

Linear state feedback controller

$$\begin{aligned} u &= l_{\text{ref}}\, r - l_1 x_1 - l_2 x_2 - \cdots - l_n x_n \\ &= l_{\text{ref}}\, r - Lx \end{aligned}$$

$$L = \begin{bmatrix} l_1 & l_2 & \dots & l_n \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Closed-Loop System

$$\begin{aligned}\dot{x} &= Ax + B(l_{\text{ref}}\, r - Lx) \\ &= \mathbf{(A - BL)}x + Bl_{\text{ref}}\, r\end{aligned}$$

$$y = Cx$$

$$Y(s) = C[sI - (A - BL)]^{-1} Bl_{\text{ref}}\, R(s)$$

We have **new system matrix**.
The characteristic polynomial is $\det[sI - (A - BL)]$.

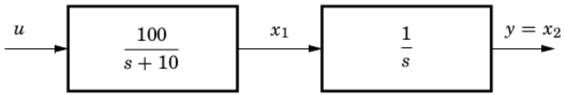
Choose L to get desired poles.

Choose l_{ref} to get $y = r$ in stationarity.

Example 1 — DC-motor

Transfer function from voltage to angle:

$$G_p(s) = \frac{b}{s(s+a)} = \frac{100}{s(s+10)}$$



Q: What are x_1 and x_2?

Example 1 — DC-motor

Transfer function from voltage to angle:

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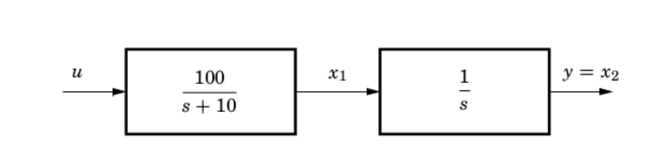

Q: What are x_1 and x_2?

State x_1 corresponds to angular speed

State x_2 corresponds to motor angle

Example 1 — DC-motor

Transfer function from voltage to angle:

$$G_p(s) = \frac{b}{s(s+a)} = \frac{100}{s(s+10)}$$

P-control: $u = K(r - y)$

$$G_o(s) = \frac{Kb}{s(s+a)}$$

$$G_t(s) = \frac{G_o(s)}{1 + G_o(s)} = \frac{bK}{s^2 + as + bK}$$

Compare the chacteristic polynomial with a desired characteristic polynomial "which we know how it behaves".

$$s^2 + as + bK = s^2 + 2\zeta\omega_0 s + \omega_0^2 \qquad \Leftrightarrow \qquad \begin{cases} \omega_0 = \sqrt{bK} \\ \zeta = \frac{0.5a}{\sqrt{bK}} \end{cases}$$

Example 1 (cont'd) — State space model

$$\begin{cases} \dot{x}_1 = -ax_1 + bu \\ \dot{x}_2 = x_1 \end{cases}$$

$$y = x_2$$

$$\begin{cases} \dot{x} = \begin{bmatrix} -a & 0 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} b \\ 0 \end{bmatrix} u \\ y = \begin{bmatrix} 0 & 1 \end{bmatrix} x \end{cases}$$

Example 1 (cont'd) — Feedback from both x_1 and x_2

Control law:

$$u = l_{\text{ref}}\, r - l_1 x_1 - l_2 x_2$$

Closed-loop system:

$$\begin{cases} \dot{x} = \begin{bmatrix} -a - bl_1 & -bl_2 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} bl_{ref} \\ 0 \end{bmatrix} r \\ y = \begin{bmatrix} 0 & 1 \end{bmatrix} x \end{cases}$$

DC-motor example

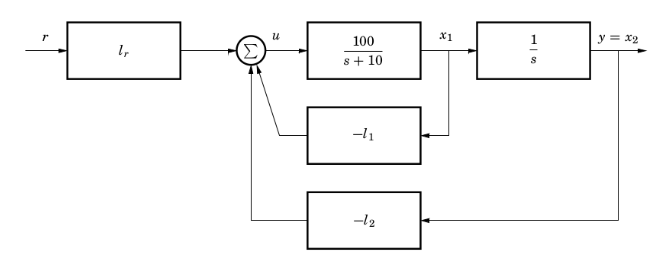

Figure 1: State feedback motor control.

The control law

$$u = l_r r - l_1 x_1 - l_2 x_2 = l_r r - Lx$$

yields the closed-loop system

$$\dot{x} = \begin{bmatrix} -10 - 100 l_1 & -100 l_2 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 100 l_r \\ 0 \end{bmatrix} r$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

Example 1 (cont'd)

Characteristic polynomial:

$$\begin{aligned} \det(sI - A) &= \begin{vmatrix} s + a + bl_1 & bl_2 \\ -1 & s \end{vmatrix} \\ &= (s + a + bl_1)s + bl_2 \\ &= s^2 + (a + bl_1)s + bl_2 \end{aligned}$$

The poles can be placed anywhere want by choosing l_1, l_2.

At stationarity:

$$0 = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -(a + bl_1)x_1 - bl_2 x_2 + bl_{\text{ref}}\, r \\ x_1 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} x_1 = 0 \\ l_2 x_2 = l_{\text{ref}}\, r \end{cases}$$

Choose $l_{\text{ref}} = l_2$. This gives $x_2 = y = r$ in stationarity.

Example 2

Can we choose characteristic polynomial (poles) freely how we want?

$$\begin{cases} \dot{x}_1 = -x_1 + u \\ \dot{x}_2 = -2x_2 \end{cases}$$

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$\begin{aligned} \det(sI - A + BL) &= \begin{vmatrix} s + 1 + l_1 & l_2 \\ 0 & s + 2 \end{vmatrix} \\ &= (s + 1 + l_1)(s + 2) \end{aligned}$$

We **can not affect** x_2!

Controllability

The system

$$\dot{x} = Ax + Bu$$

is called **controllable** if for any a and b there exist a control signal u which transfers from state $x(0) = a$ to state $x(t) = b$.

NOTE! Controllability does NOT concern y, C or D!

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-s)}Bu(s)ds$$

Cayley-Hamilton:

$$0 = A^n + a_1 A^{n-1} + \cdots + a_{n-1}A + a_n$$

where $\det(sI - A) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$.

Thus, we have

$$\begin{aligned} e^{At} &= I + At + \frac{1}{2}A^2t^2 + \cdots \\ &= \alpha_0(t)I + \alpha_1(t)A + \cdots + \alpha_{n-1}(t)A^{n-1} \end{aligned}$$

It follows that

$$\begin{aligned} x(t) &= e^{At}a + \int_0^t e^{A(t-s)}Bu(s)ds \\ &= e^{At}a + \sum_{k=0}^{n-1} \beta_k A^k B \end{aligned}$$

where $\beta_k = \int_0^t \alpha_k(t-s)u(s)ds$.

Solutions for all a and $b = x(t)$ exist if and only if

$$B, AB, A^2B, \ldots, A^{n-1}B$$

are linearly independent.

Criteria for controllability

The system $\dot{x} = Ax + Bu$ is controllable if and only if (iff)

$$\text{rank}\ \underbrace{\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}}_{W_s} = n$$

The matrix W_s is called *the controllability matrix*.

Example 2

$$\dot{x} = Ax + Bu = \begin{bmatrix} -a & 0 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} b \\ 0 \end{bmatrix} u$$

$$\text{rank } W_s = \text{rank } \begin{bmatrix} b & -ab \\ 0 & b \end{bmatrix} = 2 \qquad \text{Controllable!}$$

$$\dot{x} = Ax + Bu = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$\text{rank } W_s = \text{rank } \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = 1 \qquad \text{Not controllable!}$$

Example 3a — Pumping to lower tank

$$\begin{cases} \dot{x}_1 = -ax_1 \\ \dot{x}_2 = ax_1 - ax_2 + bu \end{cases}$$

$$\dot{x} = \begin{bmatrix} -a & 0 \\ a & -a \end{bmatrix} x + \begin{bmatrix} 0 \\ b \end{bmatrix} u$$

$$W_s = \begin{bmatrix} 0 & 0 \\ b & -ab \end{bmatrix}$$

Not controllable!

Example 3b — Pumping to upper tank

$$\begin{cases} \dot{x}_1 = -ax_1 + bu \\ \dot{x}_2 = ax_1 - ax_2 \end{cases}$$

$$\dot{x} = \begin{bmatrix} -a & 0 \\ a & -a \end{bmatrix} x + \begin{bmatrix} b \\ 0 \end{bmatrix} u$$

$$W_s = \begin{bmatrix} b & -ab \\ 0 & ab \end{bmatrix}$$

Controllable!

Example 3c — Pumping to parallel tanks

$$\begin{cases} \dot{x}_1 = -ax_1 + bu \\ \dot{x}_2 = -ax_2 + bu \end{cases}$$

$$\dot{x} = \begin{bmatrix} -a & 0 \\ 0 & -a \end{bmatrix} x + \begin{bmatrix} b \\ b \end{bmatrix} u$$

$$W_s = \begin{bmatrix} b & -ab \\ b & -ab \end{bmatrix}$$

Not controllable!

Controllable form

$$G(s) = \frac{b_1 s^{n-1} + \dots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n}$$

$$\dot{x} = \begin{bmatrix} -a_1 & -a_2 & \dots & -a_{n-1} & -a_n \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & & \ddots & & \\ 0 & & & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_1 & b_2 & \dots & b_{n-1} & b_n \end{bmatrix} x$$

$$W_s = \begin{bmatrix} 1 & * & \dots & * \\ 0 & 1 & & * \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

Controllable!

State feedback in canonical controllable form

$$A - BL = \begin{bmatrix} -a_1 - l_1 & -a_2 - l_2 & \dots & -a_{n-1} - l_{n-1} & -a_n - l_n \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & & \ddots & & \\ 0 & & & 1 & 0 \end{bmatrix}$$

New characteristic polynomial

$$s^n + (a_1 + l_1)s^{n-1} + (a_2 + l_2)s^{n-2} + \cdots + a_n + l_n$$

State feedback in canonical controllable form

$$A - BL = \begin{bmatrix} -a_1 - l_1 & -a_2 - l_2 & \dots & -a_{n-1} - l_{n-1} & -a_n - l_n \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & & \ddots & & \\ 0 & & & 1 & 0 \end{bmatrix}$$

New characteristic polynomial

$$s^n + (a_1 + l_1)s^{n-1} + (a_2 + l_2)s^{n-2} + \dots + a_n + l_n$$

Benefit: Easy to see how to choose $L = \begin{bmatrix} l_1 & l_2 & \dots & l_n \end{bmatrix}$

to **change from original characteristic polynomial to desired**!

Integral action

A limitation with ordinary state feedback controllers is that they **lack integral action**, which consequently **may result in stationary control errors**.

Introduce an **extra state** x_i as the integral of the error.

$$x_i = \int (r - y)dt \qquad \Rightarrow \qquad \dot{x}_i = r - y = r - Cx$$

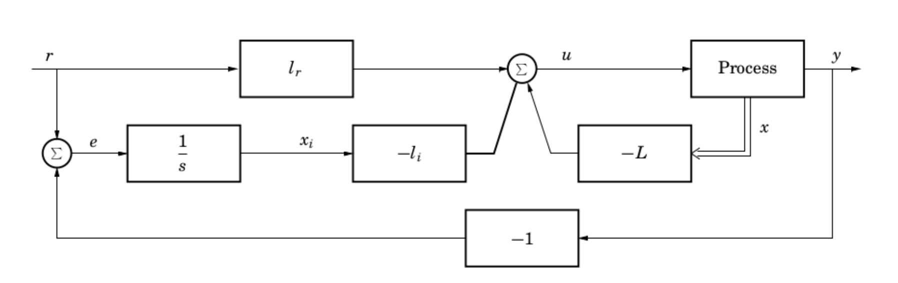

Figure 2: Introduce **extra state** x_i for state feedback with integral action.

If we **augment** the state vector x with the integral state x_i such that

$$x_e = \begin{bmatrix} \mathbf{x_1} \\ \vdots \\ \mathbf{x_n} \\ \mathbf{x_i} \end{bmatrix}$$

the augmented system can be written

$$\dot{x}_e = \begin{bmatrix} \dot{x} \\ \dot{x}_i \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} x_e + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r = A_e x_e + B_e u + B_r r$$

$$y = \begin{bmatrix} C & 0 \end{bmatrix} x_e = C_e x_e$$

If we **augment** the state vector x with the integral state x_i such that

$$x_e = \begin{bmatrix} \mathbf{x_1} \\ \vdots \\ \mathbf{x_n} \\ \mathbf{x_i} \end{bmatrix}$$

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$$y = \begin{bmatrix} C & 0 \end{bmatrix} x_e = C_e x_e$$

We have hence augmented the state-space system with a state which represents the integral of the control error and thus arrived at a **controller with integral action**. In stationarity it holds that $\dot{x}_e = 0$ and thereby that $\dot{x}_i = r - y = 0$.

State feedback with Integral Action (cont'd)

The controller now becomes

$$u = l_r r - Lx - l_i x_i = l_r r - L_e x_e$$

where

$$L_e = \begin{bmatrix} L & l_i \end{bmatrix}$$

This yields the following closed-loop state-space equations

$$\dot{x}_e = (A_e - B_e L_e)x_e + (B_e l_r + B_r)r$$

$$y = C_e x_e$$

The parameters in **the vector L_e are chosen**[1] so that we obtain a desired closed-loop pole placement, just as previously. Here the poles are given by the characteristic polynomial

$$\det(sI - (A_e - B_e L_e))$$

[1] NOT same values for L with and without integral action!!

Remark: We no longer need the parameter l_r in order to achieve $y = r$ in stationarity.

The parameter does not affect the poles of the closed-loop system, only its zeros. However, it can therefore be chosen so that the system obtains desired transient properties at setpoint changes.

We shall come back to zero placement in a later lecture.

Summary

1. Controllable form
2. State feedback control
3. Example
4. Controllability
5. Integral Action

Next lecture

- Observability
- State estimation
- Output feedback
- Pole-Zero cancellations (Warning!!)