State Space Models, Linearization, Transfer Function

Automatic Control, Basic Course, Lecture 2

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Content

From Lecture 1

- PID-control
- State-space model of plant

State Space Models

State Space Models

Consider a linear differential equation of order **n**

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_n y = b_0 \frac{d^n u}{dt^n} + b_1 \frac{d^{n-1} u}{dt^{n-1}} + \ldots + b_n u$$

For linear systems **the superposition principle** holds:

$$u = u_1 \Longrightarrow y = y_1 \text{ and}$$

$$u = u_2 \Longrightarrow y = y_2 \text{ implies}$$

$$u = c_1 \cdot u_1 + c_2 \cdot u_2 \Longrightarrow y = c_1 \cdot y_1 + c_2 \cdot y_2$$

and vice versa; We can consider the output from a sum of signals by considering the influence from each component.

State Space Models

Consider a linear differential equation of order **n**

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and vice versa; We can consider the output from a sum of signals by considering the influence from each component.

Q: Why is this not true for nonlinear systems? Example?

State Space Models

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An **alternative** to ONE differential quation of order n^{th} is to write it as a system of n **coupled differential equations, each or order one**.

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An **alternative** to <u>ONE</u> differential quation of <u>order</u> n^{th} is to write it as a system of n **coupled differential equations, each or order one**.

General State space representation:

$$\begin{cases} \dot{x}_1 & = f_1(x_1, x_2, \ldots x_n, u) \\ \dot{x}_2 & = f_2(x_1, x_2, \ldots x_n, u) \\ & \ldots \\ \dot{x}_n & = f_n(x_1, x_2, \ldots x_n, u) \\ y & = g(x_1, x_2, \ldots x_n, u) \end{cases}$$

The last row is a static equation relating the introduced **states** (x) with the input u, and the output y.

State Space Models

Consider a linear differential equation of order n

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_n y = b_0 \frac{d^n u}{dt^n} + b_1 \frac{d^{n-1} u}{dt^{n-1}} + \ldots + b_n u$$

An **alternative** to ONE differential quation of order n^{th} is to write it as a system of n coupled differential equations, each or order one.

Linear state space representation:

$$\begin{cases} \dot{x}_1 & = a_{11}x_1 + \ldots + a_{1n}x_n + b_1 u \\ \dot{x}_2 & = a_{21}x_1 + \ldots + a_{2n}x_n + b_n u \\ & \ldots \\ \dot{x}_n & = a_{n1}x_1 + \ldots + a_{nn}x_n + b_n u \\ y & = c_1 x_1 + c_2 x_2 + \ldots + c_n x_2 + du \end{cases}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ & & & \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \\ x_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \\ b_n \end{bmatrix} u$$

$$y = \begin{bmatrix} c_1 & c_2 & \ldots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \\ x_n \end{bmatrix} + du$$

State Space Models

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$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \\ x_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \\ b_n \end{bmatrix} u$$

$$y = \begin{bmatrix} c_1 & c_2 & \ldots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \\ x_n \end{bmatrix} + du$$

NOTE: **Only states (x) and inputs (u) are allowed** on the right hand side in Eq.-system above (in f and g) for it to be called a state-space representation!

State Space Models

Linear dynamics can be described in the following form

$$\dot{x} = Ax + Bu$$
$$y = Cx \, (+Du)$$

Here $x \in \mathbb{R}^n$ is a vector with states. States can have a physical ”interpretation”, but not necessary.

In this course $u \in \mathbb{R}$ and $y \in \mathbb{R}$ will be scalars.

(For MIMO systems, see Multivariable Control (FRTN10))

Example

The position of a mass m controlled by a force u is described by

$$m\ddot{x} = u$$

where x is the position of the mass.

Introduce the states $x_1 = \dot{x}$ and $x_2 = x$ and write the system on state space form. Let the position be the output.

Dynamical Systems

	Continous Time	Discrete Time (sampled)
Linear	This course	Real-Time Systems (FRTN01) / Signal proc. .
Nonlinear	Nonlinear Control and Servo Systems (FRTN05)	

Linearization

Linearization - Why?

Many systems are nonlinear. However, one can approximate them with linear ones. This to get a system that is easier to analyze.

A few examples of nonlinear systems:

- Water tanks (Labs 1,2)
- Air resistance
- Action potentials in neurons
- Pendulum under the influence of gravity
- ...

Linearization - How?

Given a nonlinear system $\dot{x} = f(x, u),\ y = g(x, u)$

Linearization - How?

Given a nonlinear system $\dot{x} = f(x, u)$, $y = g(x, u)$

1. Determine a **stationary point** (x_0, u_0) to linearize around

$$\dot{x}_0 = 0 \quad \Leftrightarrow \quad f(x_0, u_0) = 0$$

Linearization - How?

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1. Determine a **stationary point** (x_0, u_0) to linearize around

$$\dot{x}_0 = 0 \quad \Leftrightarrow \quad f(x_0, u_0) = 0$$

2. Make a first order **Taylor series expansions** of f and g around (x_0, u_0):

$$f(x, u) \approx f(x_0, u_0) + \frac{\partial}{\partial x} f(x_0, u_0)(x - x_0) + \frac{\partial}{\partial u} f(x_0, u_0)(u - u_0)$$
$$g(x, u) \approx g(x_0, u_0) + \frac{\partial}{\partial x} g(x_0, u_0)(x - x_0) + \frac{\partial}{\partial u} g(x_0, u_0)(u - u_0)$$

Notice that $f(x_0, u_0) = 0$ and let $y_0 = g(x_0, u_0)$

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Notice that $f(x_0, u_0) = 0$ and let $y_0 = g(x_0, u_0)$

3. **Introduce** $\Delta x = x - x_0$, $\Delta u = u - u_0$ and $\Delta y = y - y_0$

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   Notice that $f(x_0,u_0) = 0$ and let $y_0 = g(x_0,u_0)$

3. **Introduce** $\Delta x = x - x_0$, $\Delta u = u - u_0$ and $\Delta y = y - y_0$
4. The state-space equations in the new variables are given by:

$$\dot{\Delta x} = \dot{x} - \dot{x}_0 = f(x,u) \approx \frac{\partial}{\partial x} f(x_0,u_0)\Delta x + \frac{\partial}{\partial u} f(x_0,u_0)\Delta u = A\Delta x + B\Delta u$$

$$\Delta y = g(x,u) - y_0 \approx \frac{\partial}{\partial x} g(x_0,u_0)\Delta x + \frac{\partial}{\partial u} g(x_0,u_0)\Delta u = C\Delta x + D\Delta u$$

Example - Linearization

Example

The dynamics of a specific system is described by

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{x_2^4}{x_1^2} + x_1 + \sqrt{u+1}$$
$$y = x_1^2 + u^2$$

a) Find all stationary points
b) Linearize the system around the stationary point corresponding to $u_0 = 3$

The dynamics of a specific system is described by

$$\dot{x}_1 = x_2 \qquad = f_1(x_1,\, x_2,\, u)$$

$$\dot{x}_2 = -\frac{x_2^4}{x_1^2} + x_1 + \sqrt{u+1} \qquad = f_2(x_1,\, x_2,\, u)$$

$$y = x_1^2 + u^2 \qquad = g(x_1,\, x_2,\, u)$$

(a) Find stationary point for $u_0 = 3$: $(\dot{x}_1 = \dot{x}_2 = 0)$

$$0 = x_2$$

$$0 = -\frac{x_2^4}{x_1^2} + x_1 + \sqrt{3+1}$$

$$y = x_1^2 + 3^2$$

$$\Longrightarrow (x_{10},\, x_{20},\, u_0) = (-2,\, 0,\, 3)$$

$$y_0 = g(x_{10}, x_{20}, u_0) = 13$$

The dynamics of a specific system is described by

$$
\begin{aligned}
\dot{x}_1 &= x_2 && = f_1(x_1,\ x_2,\ u) \\
\dot{x}_2 &= -\frac{x_2^4}{x_1^2} + x_1 + \sqrt{u+1} && = f_2(x_1,\ x_2,\ u) \\
y &= x_1^2 + u^2 && = g(x_1,\ x_2,\ u)
\end{aligned}
$$

$$
\Longrightarrow (x_{10},\ x_{20},\ u_0) = (-2,\ 0,\ 3)
$$

$$
y_0 = g(x_{10}, x_{20}, u_0) = 13
$$

(b) Linearize around stationary point $(-2,\ 0,\ 3)$

$$
\begin{aligned}
\frac{\partial f_1}{\partial x_1} &= 0, & \frac{\partial f_1}{\partial x_2} &= 1, & \frac{\partial f_1}{\partial u} &= 0, \\
\frac{\partial f_2}{\partial x_1} &= +2\frac{x_2^4}{x_1^3} + 1, & \frac{\partial f_2}{\partial x_2} &= -4\frac{x_2^3}{x_1^2}, & \frac{\partial f_2}{\partial u} &= \frac{1}{2\sqrt{u+1}}, \\
\frac{\partial g}{\partial x_1} &= 2x_1, & \frac{\partial g}{\partial x_2} &= 0, & \frac{\partial g}{\partial u} &= 2u,
\end{aligned}
$$

The dynamics of a specific system is described by

$$
\begin{aligned}
\dot{x}_1 &= x_2 && = f_1(x_1, x_2, u) \\
\dot{x}_2 &= -\frac{x_2^4}{x_1^2} + x_1 + \sqrt{u+1} && = f_2(x_1, x_2, u) \\
y &= x_1^2 + u^2 && = g(x_1, x_2, u)
\end{aligned}
$$

$$
\implies (x_{10},\, x_{20},\, u_0) = (-2,\, 0,\, 3)
$$

$$
y_0 = g(x_{10}, x_{20}, u_0) = 13
$$

(b) Linearize around stationary point $(-2,\, 0,\, 3)$

$$
\begin{aligned}
&\frac{\partial f_1}{\partial x_1}_{|\{x_0,\, u_0\}} = 0, && \frac{\partial f_1}{\partial x_2}_{|\{x_0,\, u_0\}} = 1, && \frac{\partial f_1}{\partial u}_{|\{x_0,\, u_0\}} = 0, \\
&\frac{\partial f_2}{\partial x_1}_{|\{x_0,\, u_0\}} = 1, && \frac{\partial f_2}{\partial x_2}_{|\{x_0,\, u_0\}} = 0, && \frac{\partial f_2}{\partial u}_{|\{x_0,\, u_0\}} = \frac{1}{4}, \\
&\frac{\partial g}{\partial x_1}_{|\{x_0,\, u_0\}} = -4, && \frac{\partial g}{\partial x_2}_{|\{x_0,\, u_0\}} = 0, && \frac{\partial g}{\partial u}_{|\{x_0,\, u_0\}} = 6,
\end{aligned}
$$

The dynamics of a specific system is described by

$$\begin{aligned}
\dot{x}_1 &= x_2 && = f_1(x_1,\, x_2,\, u) \\
\dot{x}_2 &= -\frac{x_2^4}{x_1^2} + x_1 + \sqrt{u+1} && = f_2(x_1,\, x_2,\, u) \\
y &= x_1^2 + u^2 && = g(x_1,\, x_2,\, u)
\end{aligned}$$

$$\Longrightarrow ({\color{blue}x_{10}},\, {\color{blue}x_{20}}, {\color{red}u_0}) = (-2,\, 0,\, 3)$$
$$y_0 = g(x_{10}, x_{20}, u_0) = 13$$

(b) Linearize around stationary point $(-2,\, 0,\, 3)$

$$\frac{f(x,u)}{\partial x}_{|\{x_0,\, u_0\}} = A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \frac{f(x,u)}{\partial u}_{|\{x_0,\, u_0\}} = B = \begin{bmatrix} 0 \\ \frac{1}{4} \end{bmatrix}$$

$$\frac{g(x,u)}{\partial x}_{|\{x_0,\, u_0\}} = C = \begin{bmatrix} -4 & 0 \end{bmatrix} \qquad \frac{g(x,u)}{\partial u}_{|\{x_0,\, u_0\}} = D = \begin{bmatrix} 6 \end{bmatrix}$$

The dynamics of a specific system is described by

$$\dot{x}_1 = x_2 \qquad = f_1(x_1, x_2, u)$$

$$\dot{x}_2 = -\frac{x_2^4}{x_1^2} + x_1 + \sqrt{u+1} \qquad = f_2(x_1, x_2, u)$$

$$y = x_1^2 + u^2 \qquad = g(x_1, x_2, u)$$

$$\implies ({\color{blue}x_{10}}, {\color{blue}x_{20}}, {\color{red}u_0}) = (-2, 0, 3)$$

$$y_0 = g(x_{10}, x_{20}, u_0) = 13$$

Introduce

$$\Delta x_1 = x_1 - x_{10}, \qquad \Delta x_2 = x_2 - x_{20}$$

$$\Delta u = u - u_0 \qquad \Delta y = y - y_0$$

The state-space equations in the new variables are given by:

$$\begin{bmatrix} \frac{\Delta x_1}{dt} \\ \frac{\Delta x_2}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{4} \end{bmatrix} u$$

$$\Delta y = \begin{bmatrix} -4 & 0 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \begin{bmatrix} 6 \end{bmatrix} u$$

Transfer Function

Laplace Transformation

Let $f(t)$ be a function of time t, the Laplace transformation $\mathcal{L}(f(t))(s)$ is defined as

$$\mathcal{L}(f(t))(s) = F(s) = \int_0^\infty e^{-st} f(t) \mathrm{d}t$$

Example:

$$\mathcal{L}\left(\frac{\mathrm{d}f(t)}{\mathrm{d}t}\right)(s) = sF(s) - f(0)$$

Initial values helps to calculate what happens in transient phase!

Assuming that $f(0) = f'(0) = \cdots = f^{n-1}(0) = 0$ (common assumption during this course, but not always!!) it has the property that

$$\mathcal{L}\left(\frac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}\right)(s) = s^n F(s)$$

$$\mathcal{L}\left(\int_0^t f(\tau)\frac{de}{d\tau}\right)(s) = \frac{1}{s}F(s) \qquad \text{(integrator)}$$

See Collection of Formulae for a table of Laplace transformations.

Example - Transfer Function

Example

A system's dynamics is described by the differential equation

$$\ddot{y} + a_1\dot{y} + a_2 y = b_1\dot{u} + b_2 u.$$

After Laplace transformation we get

$$(s^2 + a_1 s + a_2)Y(s) = (b_1 s + b_2)U(s)$$

which can be written as

$$Y(s) = \overbrace{\frac{b_1 s + b_2}{s^2 + a_1 s + a_2}}^{G(s)} U(s) = G(s)U(s)$$

G(s) is called the transfer function of the system.

Transfer Function

Relation between control signal U(s) and output Y(s):

$$Y(s) = G(s)U(s)$$

G(s) often fraction of polynomal, i.e.,

$$G(s) = \frac{Q(s)}{P(s)}$$

Zeros of Q(s) are called zeros of the system, zeros of P(s) are called poles of the system.

The poles play a very important role for the system's behavior.

From State Space to Transfer Function

For a system on state space form

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

the transfer function is given by

$$G(s) = C(sI - A)^{-1}B + D$$

Observe: the denominator of $G(s)$ is given by $P(s) = \det(sI - A)$, so eigenvalues of A are poles of the system.

From Transfer Function to State Space

Can be done in several ways, see Collection of Formulae.

Example

A system's transfer function is

$$G(s) = \frac{2s + 1}{s^3 + 4s - 8}$$

Write the system on a state space form of your choice.

Three Ways to Describe a Dynamical System

Differential equation

$$\ddot{y} + a_1\dot{y} + a_2 y = b_1\dot{u} + b_2 u$$

$$x_1 = y$$
$$x_2 = \dot{y}$$
$$\dots$$

$$\mathcal{L}\left(\frac{d^n f(t)}{dt^n}\right)(s) = s^n F(s)$$

Collection of Formulae

State space

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

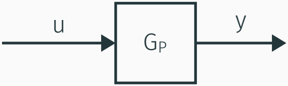
Transfer function

$$Y(s) = G(s)U(s) = \frac{Q(s)}{P(s)}U(s)$$

$$G(s) = C(sI - A)^{-1}B + D$$

Block Diagram Representation

When the blocks in a block diagram are replaced by transfer functions, it is possible to describe the relations between signals in an easy way.

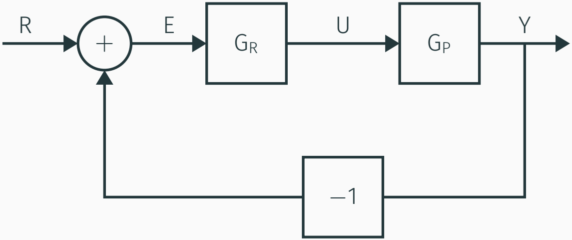


$$Y(s) = G_P(s)U(s)$$

Block Diagram - Components

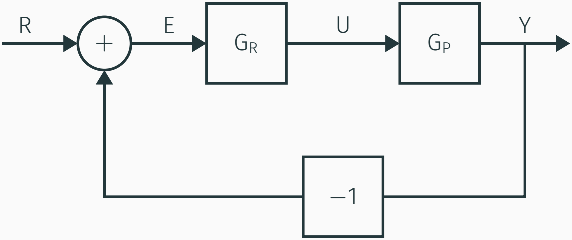
Most block diagrams consist of three components:

- Blocks - Transfer functions
- Arrows - Signals
- Summations



where R, E, U, Y are the Laplace transformations of the reference r(t), control error e(t), control signal u(t), and output y(t), respectively.

Determine Transfer Function From Block Diagram



$$Y = G_P U, \quad U = G_R E, \quad E = R - Y$$

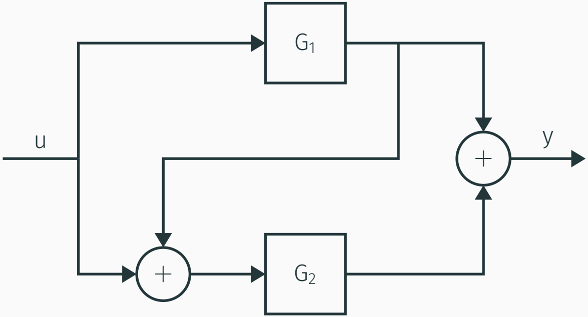
From the equations above the transfer function between r and y is

$$Y = \frac{G_P G_R}{1 + G_P G_R} R$$

Example - Transfer Functions

Example

Two systems, G_1 and G_2, are interconnected as in the figure below



Compute the transfer function from u to y, G_{yu}.

Summary

This lecture

1. State Space Models
2. Linearization
3. Transfer Function
4. Block Diagram Representation

Next lecture

- Impulse Response Analysis
- Step Response Analysis