Systems Engineering/Process control L6

- Linearization
- Feedback systems – an example

Reading: *Systems Engineering and Process Control:* 6.1–6.2

Equilibrium points

- Nonlinear process on state-space form:

$$\frac{dx}{dt} = f(x,u)$$
$$y = g(x,u)$$

- Process equilibria/stationary points: all points (x^0, u^0) with

$$f(x^0, u^0) = 0$$

that is, all state time derivatives are zero

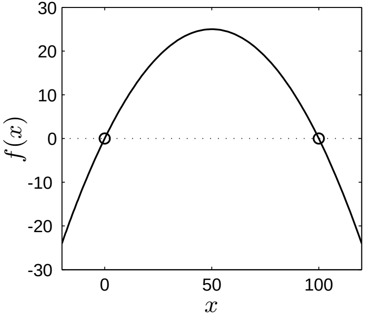
Example: Logistic growth model

- Suppose *birth rate* $r = 1$, *carrying capacity* $k = 100$:

$$\frac{dx}{dt} = x\left(1 - \frac{x}{100}\right) = f(x)$$

- Equilibrium points ($f(x_0) = 0$):

$$x^0\left(1 - \frac{x^0}{100}\right) = 0 \quad \Rightarrow \quad \begin{cases} x^0 = 0 \\ x^0 = 100 \end{cases}$$



Linearization – one variable

Suppose nonlinear system with one (scalar) variable: $\dot{x} = f(x)$

1. Find stationary point x^0 with $f(x^0) = 0$
2. Approximate $f(x)$ with a straight line through x^0:

$$f(x) \approx \underbrace{f(x^0)}_{=0} + \underbrace{\frac{df}{dx}(x^0)}_{a}(x - x^0)$$

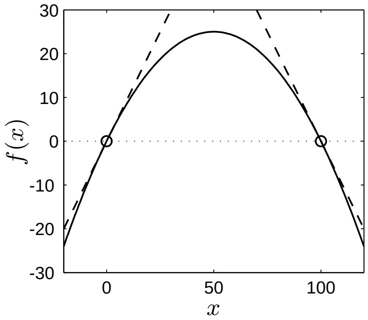
3. Change state variable to deviations Δx from stationary point:

$$\Delta x = x - x^0$$

4. System can then be written on the form

$$\frac{d\Delta x}{dt} \approx a\Delta x$$

Example: Logistic growth model



- Linearized models:

$$x^0 = 0 \quad \Rightarrow \quad \frac{d\Delta x}{dt} \approx \Delta x \quad \text{(locally unstable, why?)}$$

$$x^0 = 100 \Rightarrow \quad \frac{d\Delta x}{dt} \approx -\Delta x \quad \text{(locally asymptotically stable, why?)}$$

- For what x values can each linearized model be used?

Linearization – general case

1. Find a stationary point (x^0, u^0) to the system

$$\frac{dx}{dt} = f(x,u)$$
$$y = g(x,u)$$

2. Make a 1st order Taylor approximation of f and g around (x^0, u^0):

$$f(x,u) \approx \underbrace{f(x^0,u^0)}_{=0} + \frac{\partial f}{\partial x}(x^0,u^0)(x-x^0) + \frac{\partial f}{\partial u}(x^0,u^0)(u-u^0)$$

$$g(x,u) \approx \underbrace{g(x^0,u^0)}_{=y^0} + \frac{\partial g}{\partial x}(x^0,u^0)(x-x^0) + \frac{\partial g}{\partial u}(x^0,u^0)(u-u^0)$$

Linearization – general case

3. Introduce variables $\Delta x = x - x^0$, $\Delta u = u - u^0$ and $\Delta y = y - y^0$
4. The linearized system can now be written on the form

$$\frac{d\Delta x}{dt} = \frac{dx}{dt} = f(x,u) \approx \underbrace{\frac{\partial f}{\partial x}(x^0,u^0)}_{A}\Delta x + \underbrace{\frac{\partial f}{\partial u}(x^0,u^0)}_{B}\Delta u$$

$$\Delta y = g(x,u) - y^0 \quad \approx \underbrace{\frac{\partial g}{\partial x}(x^0,u^0)}_{C}\Delta x + \underbrace{\frac{\partial g}{\partial u}(x^0,u^0)}_{D}\Delta u$$

(Remember: The resulting linear state-space model approximates original nonlinear model close to the stationary point (x_0, u_0)*)*

Linearization – general case

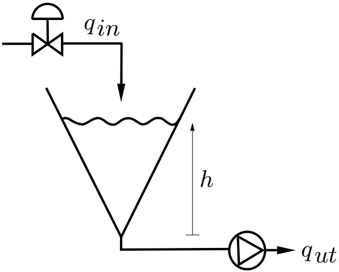
- Note that f and g can be vectors
- Example: Two states x_1 and x_2, one input u one output y:

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \; f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix},$$

$$\frac{\partial f}{\partial x} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}, \qquad \frac{\partial f}{\partial u} = \begin{pmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{pmatrix}$$

$$\frac{\partial g}{\partial x} = \begin{pmatrix} \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} \end{pmatrix}, \qquad \frac{\partial g}{\partial u} \text{ is scalar}$$

Example: Linearization of conic tank



Suppose that q_{ut} is constant. Model:

$$\frac{dh}{dt} = \frac{4}{\pi h^2}\left(q_{in} - q_{ut}\right) = f(h, q_{in})$$

$$y = h = g(h, q_{in})$$

Example: Linearization of conic tank

1. Stationary point: $f(h^0, q_{in}^0) = 0 \quad \Rightarrow \quad q_{in}^0 = q_{ut}$

2. Compute partial derivatives:

$$\frac{\partial f}{\partial h}(h^0, q_{in}^0) = -\frac{8}{\pi (h^0)^3}(q_{in}^0 - q_{ut}) = 0 \qquad \frac{\partial f}{\partial q_{in}}(h^0, q_{in}^0) = \frac{4}{\pi (h^0)^2}$$

$$\frac{\partial g}{\partial h}(h^0, q_{in}^0) = 1 \qquad \frac{\partial g}{\partial q_{in}}(h^0, q_{in}^0) = 0$$

3. New variables: $\Delta h = h - h^0$, $\Delta q_{in} = q_{in} - q_{in}^0$, $\Delta y = y - y^0$

4. Linear state-space model:

$$\frac{d\Delta h}{dt} \approx \frac{4}{\pi (h^0)^2} \Delta q_{in}$$

$$\Delta y = \Delta h$$

Example: Linearization of conic tank

1. Linear state-space model:

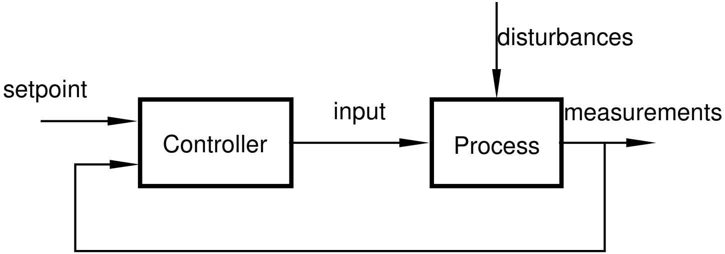
$$\frac{d\Delta h}{dt} \approx \frac{4}{\pi(h^0)^2}\,\Delta q_{in}$$
$$\Delta y = \Delta h$$

   approximates nonlinear model around h_0

2. Is the system:
   - Locally stable
   - Locally asymptotically stable
   - Locally marginally stable
   - Locally unstable

   Why?

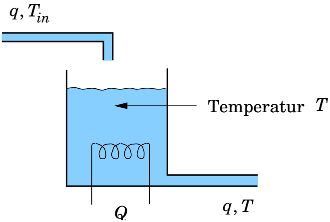
Feedback systems – closed loop control



Controller should be designed such that:

- the closed loop system is stable
- measurement signal follows setpoint (servo problem)
- disturbances are eliminated (control problem)
- the system is insensitive to model errors and parameter variations
- too much measurement noise is not feed back to process

Example: Temperature control



- Energy balance:

$$V\rho C_p \frac{dT(t)}{dt} = q\rho C_p (T_{in}(t) - T(t)) + Q(t)$$

- Let $K_1 = \frac{1}{q\rho C_p}$, $T_1 = \frac{V}{q}$:

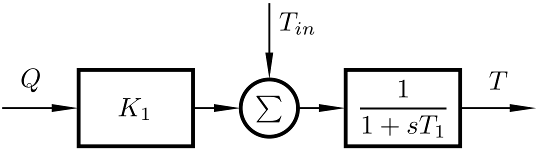
$$T_1 \frac{dT(t)}{dt} + T = K_1 Q(t) + T_{in}(t)$$

- Laplace transform:

$$T(s) = \frac{1}{1+sT_1}\Big(K_1 Q(s) + T_{in}(s)\Big)$$

- Linear model: T, Q, T_{in} denote deviations from a stationary point
- Objective: Keep $T = T_{ref}$ using Q despite variations in T_{in}

Open-loop system



Response to step disturbance $T_{in} = 1$ (suppose $Q = 0$):

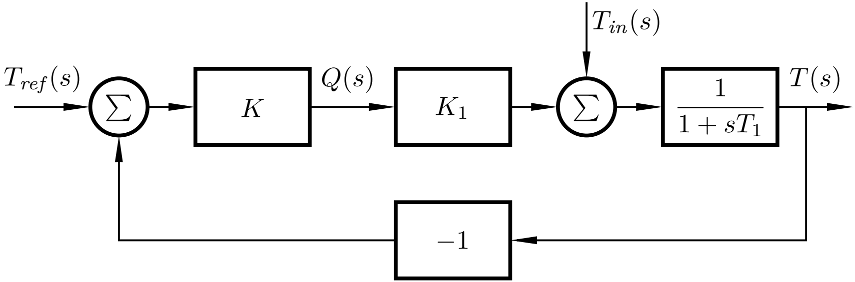
$$T(s) = \frac{1}{1+sT_1} T_{in}(s) = \frac{1}{s(1+sT_1)}$$

$$T(t) = 1 - e^{-t/T_1}$$

Stationarity: $T = 1$

P control

$$Q(t) = K\big(T_{ref}(t) - T(t)\big) = Ke(t)$$
$$Q(s) = K\big(T_{ref}(s) - T(s)\big) = KE(s)$$



P control

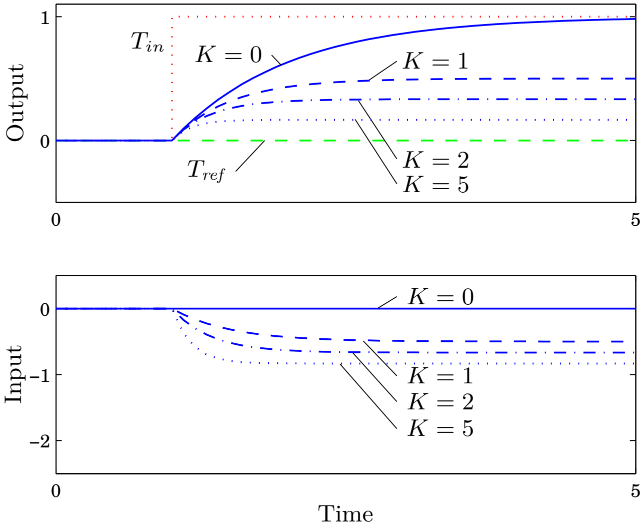
$$T(s) = \frac{1}{1 + sT_1}\Big(K_1 K\big(T_{ref}(s) - T(s)\big) + T_{in}(s)\Big)$$

$$T(s) = \frac{K_1 K}{1 + sT_1 + K_1 K} T_{ref}(s) + \frac{1}{1 + sT_1 + K_1 K} T_{in}(s)$$

- Pole:
$$s = -\frac{1 + K_1 K}{T_1}$$
- Asymptotically stable if $K > -1/K_1$
- Stationarity ($s = 0$): $T = \dfrac{K_1 K}{1 + K_1 K} T_{ref} + \dfrac{1}{1 + K_1 K} T_{in}$

Simulation of step disturbance

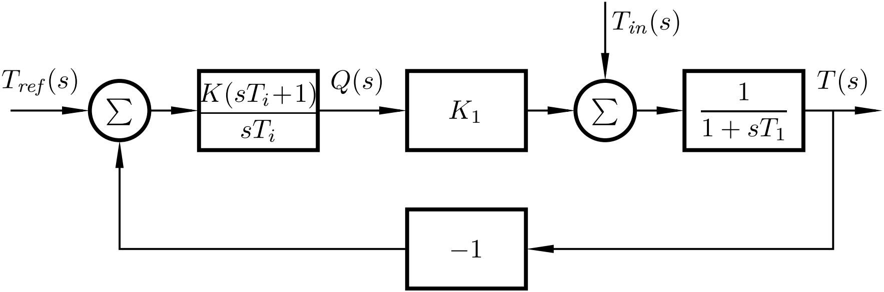
Parameters: $T_1 = K_1 = 1$:



PI control

$$Q(t) = K\left(e(t) + \frac{1}{T_i}\int^t e(\tau)\,d\tau\right)$$

$$Q(s) = K\left(1 + \frac{1}{sT_i}\right)E(s)$$



PI control

$$T(s) = \frac{1}{1+sT_1}\left(\frac{K_1K(sT_i+1)}{sT_i}\Big(T_{ref}(s) - T(s)\Big) + T_{in}(s)\right)$$

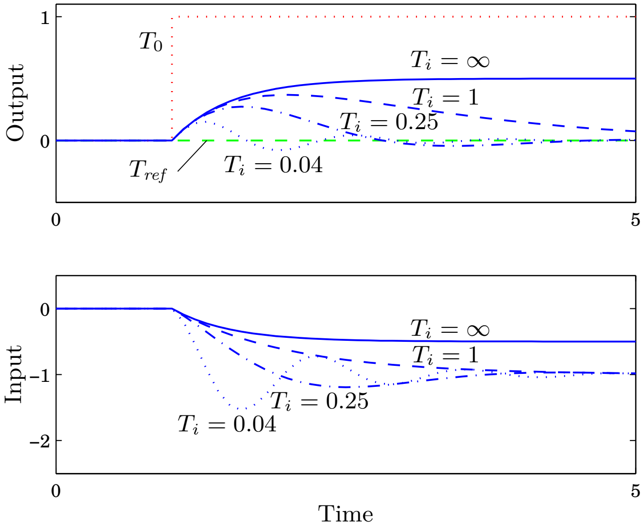
$$T(s) = \frac{K_1K(sT_i+1)}{s^2T_1T_i + s(K_1K+1)T_i + K_1K}T_{ref}(s)$$

$$+\frac{sT_i}{s^2T_1T_i + s(K_1K+1)T_i + K_1K}T_{in}(s)$$

- Asymptotically stable if $K > -1/K_1$ and $KT_i > 0$
- Stationarity ($s = 0$): $T = T_{ref}$

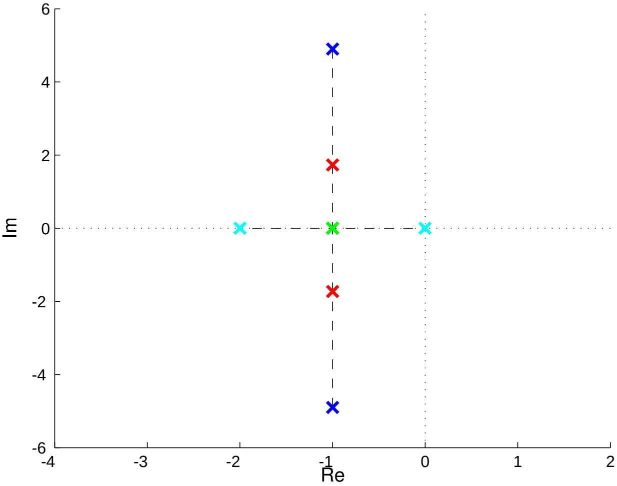
Simulation of step disturbance

Parameters: $T_1 = K_1 = K = 1$:



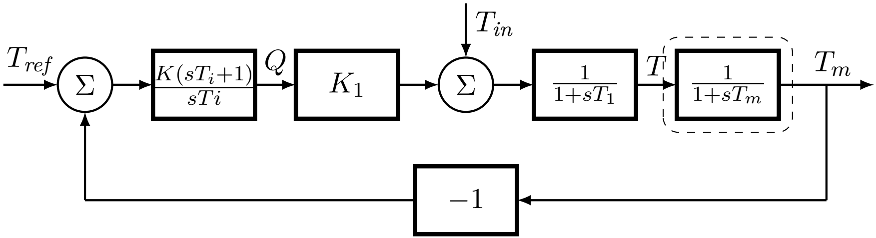
Feedback system poles

×: $T_i = 0.04$ ×: $T_i = 0.25$
×: $T_i = 1$ ×: $T_i \to \infty$



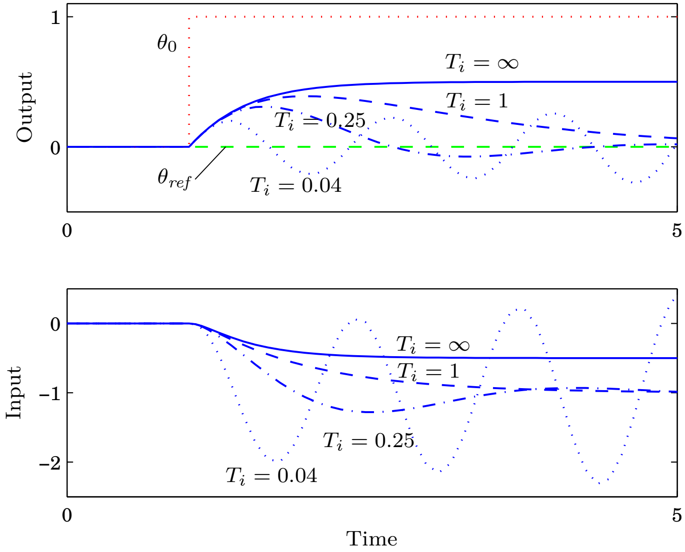
Sensitivity to unmodeled dynamics

- Suppose sensor with dynamics $T_m(s) = \dfrac{1}{1+sT_m} T(s)$
- Closed loop system becomes:



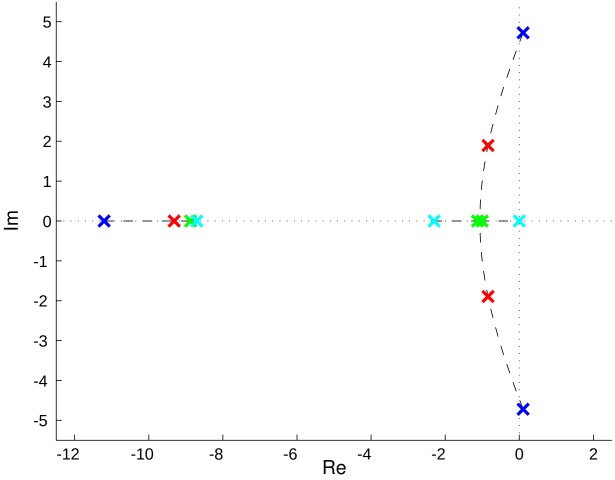
Simulation with unmodeled dynamics

Parameters: $T_m = 0.1$, $T_1 = K_1 = K = 1$



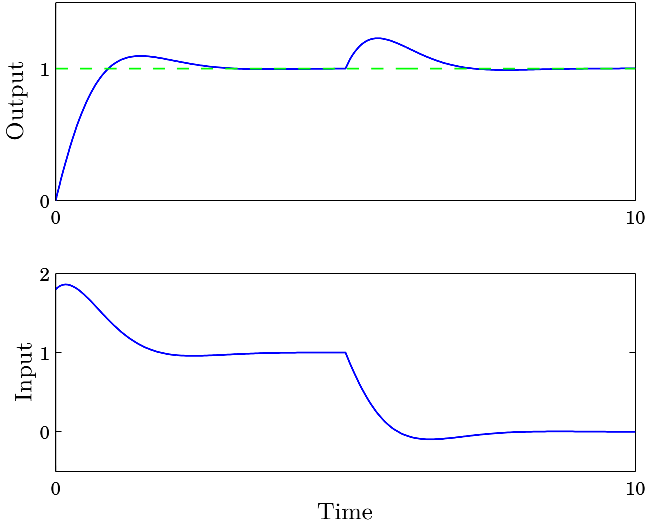
Poles for feedback system with sensor dynamics

×: $T_i = 0.04$ ×: $T_i = 0.25$
×: $T_i = 1$ ×: $T_i \to \infty$



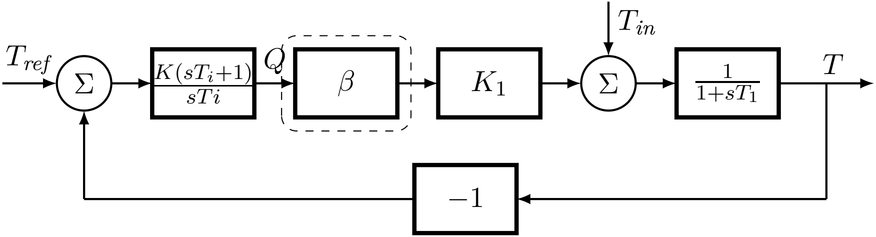
Simulation of setpoint change, disturbance

Parameters: $T = K_1 = 1,\ K = 1.8,\ T_i = 0.45$:



Sensitivity to parameter variations

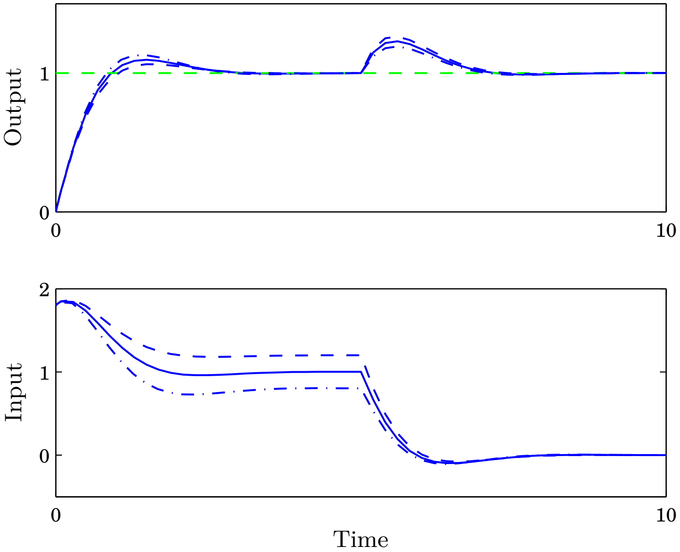
- Suppose the gain model K_1 for Q is wrong
- We get:



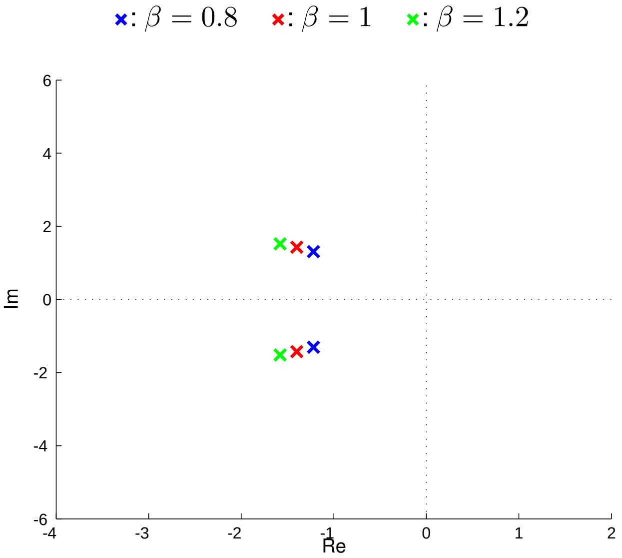
- with $\beta = 0.8, 1, 1.2$

Simulation with wrong gain

Parameters: $T = K_1 = 1, K = 1.8,\ T_i = 0.45, \quad \pm 20\%$ change in Q



Poles for feedback system with wrong gain



Conclusions from example

- We can change the system dynamics with feedback
  - Place poles of closed loop system using controller parameters (*pole placement*), we want:
    - Stability, fast and well-damped responses
- Sometimes, we get a stationary error
  - What can be said about stationary errors? – F7
- Unmodeled dynamics and parameter variations affect behavior
  - How to analyze closed loop system sensitivity? – F7