Last Week

- Laplace transform - single vs double sided
- Initial and Final Value Theorem

Initial and Final Value Theorem

Initial Value Theorem Suppose that f is causal and that the Laplace transform $F(s)$ is rational and strictly proper. Then

$$\lim_{t \to +0} f(t) = \lim_{s \to +\infty} sF(s)$$

Final Value Theorem. Suppose that f is causal with rational Laplace transform $F(s)$. If all poles of $sF(s)$ have negativ real part, then

$$\lim_{t \to +\infty} f(t) = \lim_{s \to +0} sF(s)$$

Lecture 2

- **(Cauchy's) Argument Principle**
- Nyquist criterion
- Example: Trailer
- Example: Feedback with time delay
- Bode's relations between gain and phase

Argument variation

Let Γ be a simple closed curve in the complex plane surrounding the domain D.

The change in the argument for the complex function $F(s)$ when s follows the boundary to D (i.e., follows Γ) in a counter-clockwise (CCW) direction, is called the argument variation of F along Γ and is denoted $\Delta_\Gamma \arg F$:

$$\Delta_\Gamma \arg F := \int_\Gamma \left(\frac{d}{ds} \arg F(s) \right) ds$$

(Cauchy's) argument principle

Suppose that $F(s)$ is analytic in a neighborhood of D except for a finite number of poles in D. Then

$$\frac{1}{2\pi} \Delta_\Gamma \arg F = N_F - P_F$$

where N_F is the number of zeros and P_F the number of poles of F in D.

Proof of the Argument Principle

The argument function is the imaginary part of the complex logarithm, so

$$\begin{aligned} \Delta_\Gamma \arg F &= \int_\Gamma \left(\frac{d}{ds} \arg F(s) \right) ds \\ &= \text{Im} \int_\Gamma \left(\frac{d}{ds} \log F(s) \right) ds = \text{Im} \int_\Gamma \frac{F'(s)}{F(s)} ds \end{aligned}$$

F'/F is singular exactly in the poles and zeros of F.

Proof cont'd

$$F(s) = \frac{(s - z_1) \cdots (s - z_{N_F})}{(s - p_1) \cdots (s - p_{P_F})} G(s)$$

where G has no poles and zeros in D. Then

$$\log F(s) = \sum_{j=1}^{N_F} \log(s - z_j) - \sum_{j=1}^{P_F} \log(s - p_j) + \log G(s)$$

Derivation and integration gives

$$\begin{aligned} \text{Im} \frac{1}{2\pi} \int_\Gamma \frac{d}{ds} \log F(s) ds &= \frac{1}{2\pi} \text{Im} \int_\Gamma \left(\sum_{j=1}^{N_F} \frac{1}{s - z_j} - \sum_{j=1}^{P_F} \frac{1}{s - p_j} + \frac{G'(s)}{G(s)} \right) ds \\ &= N_F - P_F \end{aligned}$$

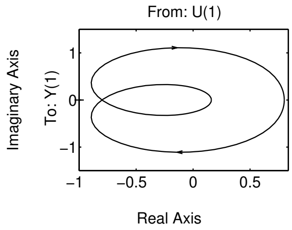
Lecture 2

- (Cauchy's) Argument Principle
- **Nyquist criterion**
- Example: Trailer
- Exemple: Feedback with time delay
- Bode's relations between gain and phase

Nyquist Criterion

Regler AK: If $L(s)$ is stable, then the closed loop system $[1+L(s)]^{-1}$ is also stable if and only if the Nyquist curve $L(i\omega)$ does not encircle -1.

More general: The difference of the number of unstable poles to $[1+L(s)]^{-1}$ and the number of unstable poles of $L(s)$ equals the number of clockwise encirclements of the point -1.

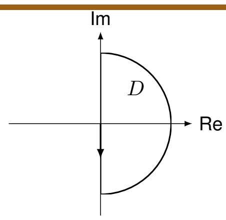


Proof of the Nyquist criterion

Apply the argument principle on

$$F(s) = 1 + L(s)$$

where D as in picture, and radius large enough to contain all poles and zeros in the RHPL.



Then

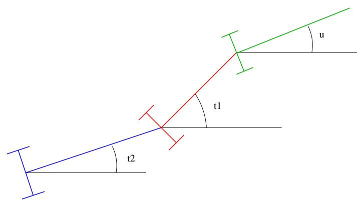
$$\begin{aligned} P_F &= \text{number of unstable poles to } 1+L(s) = P_{\text{open}} \\ N_F &= \text{number of unstable poles to } [1+L(s)]^{-1} = P_{\text{closed}} \\ \frac{1}{2\pi}\Delta_\Gamma \arg F &= \text{number of CCW encirclements of } 0 \text{ by } F(s) \\ & \quad \text{when } s \text{ moves around boundary of } D \text{ CCW} \\ &= \text{nr of clockwise encircl. of -1 of Nyquist curve } L(i\omega) \\ & \quad \text{(direction is opposite, goes from } -\infty \text{ to } \infty) \end{aligned}$$

$$P_{\text{closed}} - P_{\text{open}} = \text{nr of clockwise encirclements around -1 of } L(i\omega)$$

Lecture 2

- (Cauchy's) Argument Principle
- Nyquist criterion
- **Example: Trailer**
- Example: Feedback with time delay
- Bode's relations between gain and phase

Example: Trailer



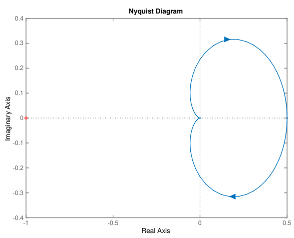
When trailer moves forward with speed $v = 1$:

$$Y(s) = \underbrace{\frac{1}{(s+2)(s+1)}}_{G(s)} U(s)$$

Example: Trailer moving forward with P-control

P-control: $U(s) = -kY(s)$. Gives $L = kG$.

```
s = tf('s')
G = 1/((s+2)*(s+1))
nyquist(G)
```

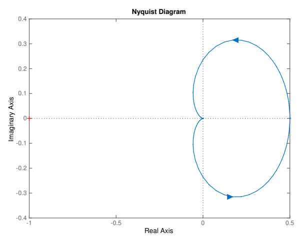


Stable if $L(i\omega) = k\frac{1}{(i\omega+2)(i\omega+1)}$ does not encircle -1.
True for all $k > 0$ (and some $k < 0$)

Example: Trailer moving backwards with P-control

Now $G(s) = \frac{1}{(s-2)(s-1)}$

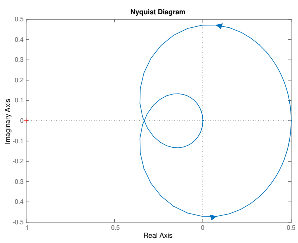
```
G = 1/((s-2)*(s-1))
nyquist(G)
```



When does $L(i\omega) = k\frac{1}{(i\omega-2)(i\omega-1)}$ encircle -1 two times counter-clockwise?
Never. So P-control can not be used.

Example: Trailer moving backwards with PD-control

Lets try this PD-controller: $U(s) = -k(1+s)Y(s)$.



Stable if $L(i\omega) = k\frac{1+i\omega}{(i\omega-2)(i\omega-1)}$ encircles -1 two times counter-clockwise.
True when $k > 3$. PD-control works

Lecture 2

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- **Example: Feedback with time delay**
- Bode's relations between gain and phase

Example: System with time delay

Is the system

$$\dot{y}(t) = y(t) - 2y(t-0.5)$$

stable?

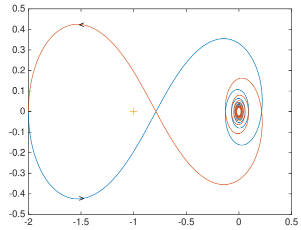
This can be viewed as a feedback system

$$\dot{y}(t) = y(t) + u(t)$$
$$u(t) = -2y(t-0.5)$$

Can use Nyquist criterion with $L = P(s)C(s) = \frac{2e^{-0.5s}}{s-1}$

Example: System with time delay

$$\dot{y}(t) = y(t) - 2y(t-0.5)$$



Stable, since $L(i\omega) = \frac{2e^{-i0.5\omega}}{i\omega-1}$ encircles -1 one time counter clock-wise.

Lecture 2

- (Cauchy's) Argument Principle
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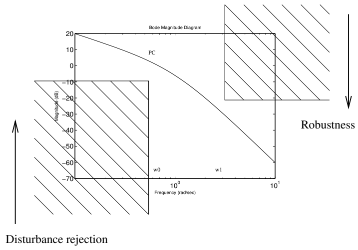
- Example: Trailer
- Example: Feedback with time delay
- **Bode's relations between gain and phase**

Design tradeoffs

A control system should typically have high gain $|P(i\omega)C(i\omega)|$ at low frequencies to reduce impact of disturbances and to follow the reference signal r, but low gain at high frequencies to avoid stability problems and the effect of measurement noise



How fast can one go from high gain to low gain for different frequencies?

Bode's relations — Approximative version

If $G(s)$ is stable and has no zeros in the RHPL and no time delay then

$$\arg G(i\omega_0) \approx \frac{\pi}{2} \frac{d \log |G(i\omega)|}{d \log \omega}\Big|_{\omega=\omega_0}$$

If there are zeros in the RHPL or time delay the phase will be smaller

Conclusion: The slope of the amplitude determines the phase.

Phase -180 degree corresponds to slope -2 (with log-log scales)

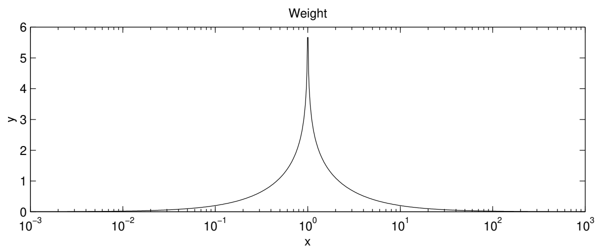
At the cut off frequency (where the amplitude equals one) the slope needs to be > -2 (around -1.5 is recommended). Otherwise the Nyquist curve will go the wrong way around -1

Can not reduce loop gain too fast.

Bode's relation(s) — Exact version

If $G(s)$ is stable and minimum phase (no zeros in RHPL or time delays) then

$$\arg G(i\omega_0) = \frac{1}{\pi}\int_0^\infty \frac{d \log |G(i\omega)|}{d \log \omega} \underbrace{\log \left|\frac{\omega+\omega_0}{\omega-\omega_0}\right|}_{\text{weight function}} d \log \omega$$



Lecture 2

- (Cauchy's) Argument Principle
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Bode's relation – Proof

- We first show

$$\arg G(i\omega_0) = \frac{2\omega_0}{\pi}\int_0^\infty \frac{\log |G(i\omega)| - \log |G(i\omega_0)|}{\omega^2 - \omega_0^2} d\omega$$

- Changes of variables and partial integration give

$$\arg G(i\omega_0) = \frac{1}{\pi}\int_0^\infty \frac{d \log |G(i\omega)|}{d \log \omega} \underbrace{\log \left|\frac{\omega+\omega_0}{\omega-\omega_0}\right|}_{\text{weight function}} d \log \omega$$

Bode's relation – Proof cont'd

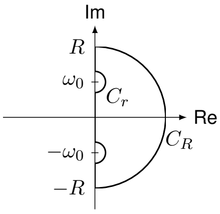
Let C be the depicted curve, then

$$\int_C \frac{\log G(s) - \log |G(i\omega_0)|}{s^2 + \omega_0^2} ds = 0$$

since the function is analytic on and inside C

Integal over C satisfies:

$$\int_C = \int_{C_r} + \int_{C_r} + \int_{-iR}^{iR} + \int_{C_R} = 0$$



Bode's relation – Proof cont'd

- Integral on C_R: $\int_{C_r} \to 0$ as $R \to \infty$ (proper)
- Integral on (both) C_r ($r \to 0$):

$$\int_{C_r} \frac{\log G(s) - \log |G(i\omega_0)|}{s^2 + \omega_0^2} ds = \int_{C_r} \frac{\log G(s) - \log |G(i\omega_0)|}{(s - i\omega_0)(s + i\omega_0)} ds$$

$$= \int_{C_r} \frac{\log(|G(s)| e^{i \arg G(s)} / |G(i\omega_0)|)}{(s - i\omega_0)(s + i\omega_0)} ds$$

$$\overset{s \to i\omega_0}{=} \frac{1}{2i\omega_0} \int_{C_r} \frac{\log(e^{i \arg G(i\omega_0)})}{s - i\omega_0} ds$$

$$= \frac{i \arg G(i\omega_0)}{2i\omega_0} \int_{C_r} \frac{1}{s - i\omega_0} ds = \frac{i \arg G(i\omega_0)}{2\omega_0} \pi$$

- Therefore, when $R \to \infty$ and $r \to 0$:

$$\frac{i \arg G(i\omega_0)}{\omega_0} \pi = -i \int_{-\infty}^{\infty} \frac{\log G(i\omega) - \log |G(i\omega_0)|}{\omega_0^2 - \omega^2} d\omega$$

Bode's relation – Proof cont'd

- Rewrite from previous slide:

$$\arg G(i\omega_0) = -\frac{\omega_0}{\pi} \int_{-\infty}^{\infty} \frac{\log G(i\omega) - \log |G(i\omega_0)|}{\omega_0^2 - \omega^2} d\omega$$

- Since $\log G(i\omega) = \log |G(i\omega)| + i \arg(G(i\omega))$ and
  - $\log |G(i\omega)|$ is even
  - $\arg(G(i\omega))$ is odd:

$$\arg G(i\omega_0) = \frac{2\omega_0}{\pi} \int_0^{\infty} \frac{\log |G(i\omega)| - \log |G(i\omega_0)|}{\omega^2 - \omega_0^2} d\omega$$

which shows the first claim

Bode's relation – Proof cont'd

- To prove the second claim, we change variable $\omega = e^x$:

$$\int_{-\infty}^{\infty} \frac{\log |G(ie^x)| - \log |G(i\omega_0)|}{e^{2x} - \omega_0^2} e^x dx$$

$$= \int_{-\infty}^{\infty} (\log |G(ie^x)| - \log |G(i\omega_0)|) \frac{1}{e^x - \omega_0^2 e^{-x}} dx$$

- Define

$$\phi(x) = \log \frac{e^x + 1}{|e^x - 1|} \quad \text{with} \quad \frac{d}{dx}\phi(x) = -\frac{2}{e^x - e^{-x}}$$

then

$$\frac{d}{dx}\phi(x - \log \omega_0) = -\frac{2}{e^{x - \log \omega_0} - e^{-x + \log \omega_0}}$$

$$= -\frac{2}{e^x/\omega_0 - e^{-x}\omega_0} = -\frac{2\omega_0}{e^x - \omega_0^2 e^{-x}}$$

Bode's relation – Proof cont'd

- Partial integration gives

$$\frac{2\omega_0}{\pi} \int_{-\infty}^{\infty} (\log |G(ie^x)| - \log |G(i\omega_0)|) \frac{1}{e^x - \omega_0^2 e^{-x}} dx$$

$$= -\frac{1}{\pi} \int_{-\infty}^{\infty} (\log |G(ie^x)| - \log |G(i\omega_0)|) \frac{d}{dx}\phi(x - \log \omega_0) dx$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d \log |G(ie^x)|}{dx} \phi(x - \log \omega_0) dx$$

$$- [(\log |G(ie^x)| - \log |G(i\omega_0)|)\phi(x - \log \omega_0)]_{x \to -\infty}^{x \to \infty}$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d \log |G(ie^x)|}{dx} \phi(x - \log \omega_0) dx$$

- Changing variables back, $x = \log w$, gives:

$$\arg G(i\omega_0) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d \log |G(i\omega)|}{d \log \omega} \left| \frac{e^{\log \omega - \log \omega_0} + 1}{e^{\log \omega - \log \omega_0} - 1} \right| d \log \omega$$

which is readily rewritten to Bode's relation

Hint to problem 1c

If one first determines $Y(s)$ one can then have use of the fact that for any complex number v we have the identity

$$(sI - A)^{-1}(s - v)^{-1} = -(sI - A)^{-1}(vI - A)^{-1} + (vI - A)^{-1}(s - v)^{-1}.$$

(If you use this identity, you should prove it!) Apply with $v = i\omega$ and $v = -i\omega$, combine the results and do inverse laplace.

Also remember that $\text{Im}(z) = (z - \bar{z})/(2i)$ and $\sin \omega t = \text{Im}(e^{i\omega t})$ and $\mathcal{L}(e^{tA}) = (sI - A)^{-1}$