Last week

- State Space Realizations (pp 139-150)
- $G(s)$, denominator and numerator, poles and zeros
- Change of coordinates, diagonal and controllable form
- State-feedback
- Observers
- Feedback from estimated states
- Integral action by disturbance model

Lecture 5

- Controllability – Existence of control signal
  - Which state directions can be controlled ?
- Observability – Determine state
  - Which state directions can not be seen?
- Kalman's decomposition theorem
- Cancelled dynamics <=> lack of controllability or observability

Controllability

How should **controllability** be defined ?

Some (not used) alternatives:

By proper choice of control signal u

- any state x_0 can be made an equilibrium
- any state trajectory $x(t)$ can be obtained
- any output trajectory $y(t)$ can be obtained

The most fruitful definition has instead turned out to be the following

Controllability

The state equation

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0$$

is called *controllable* if for any x_0 and $T > 0$, there exists $u(t)$ such that $x(T) = 0$ ("Controllable to origin")

Question: Is this equivalent to the following definition:

"for $x_0 = 0$ and any x_1 and $T > 0$, there exists $u(t)$ such that $x(T) = x_1$" ("Controllable from origin")

The audience is thinking!

Hint: $x(T) = e^{AT}x_0 + \int_0^T e^{A(T-t)}Bu(t)dt$

Controllability Gramian

The matrix

$$W(T) = \int_0^T e^{-At} B B^T e^{-A^T t} dt$$

is called the *controllability Gramian*.

Note that it is positive semidefinite, $W(T) \geq 0$

The main controllability result is the following

Theorem Controllability Test

The following conditions are equivalent:

(i) The system $\dot{x}(t) = Ax(t) + Bu(t)$ is controllable.

(ii) $\operatorname{rank}[B\ AB\ A^2B\ \ldots\ A^{n-1}B] = n$.

(iii) $W(T)$ is invertible for any $T > 0$

(iv) For any $\lambda \in \mathbf{C}$ we have rank$[A - \lambda I\ \ B] = n$

We will prove $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$

The condition (iv), not proved here, is called the PBH test (Popov-Belevitch-Hautus).

Analysing controllability

The system is by definition controllable iff we for any x_0 and T can find control signal $u(t), t \in [0, T]$ that solves (see hint some slides above)

$$-x_0 = \int_0^T e^{-At} B u(t) dt \qquad (\star)$$

Cayley-Hamilton's theorem (google it) says that A^k for $k \geq n$ can be written as a linear combination of $I, A, A^2, \dots A^{n-1}$, so

$$e^{-At} = \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} A^k = \sum_{k=0}^{n-1} f_k(t) A^k, \quad (\text{for some } f_k(t)).$$

Therefore the condition ($\star$) can be written

$$-x_0 = [B \ AB \ A^2 B \ \dots \ A^{n-1} B] F(u), \qquad (\star\star)$$

for some vector $F(u)$ with elements $F_k(u) = \int_0^T f_k(t) u(t) dt$

Proof of $(i) \Rightarrow (ii)$

Proof by contradiction: Assume (ii) does not hold, i.e. the controllability matrix does not have full rank.

This means there is a vector, lets call it $-x_0$, that is not in the column span of

$$[B\ AB\ A^2B\ \dots\ A^{n-1}B]$$

This contradicts ($\star\star$), so (i) does not hold.

Proof of $(ii) \Rightarrow (iii)$

Assume (iii) does not hold. Then there is a $p \neq 0$ so $W(T)p = 0$.

$$0 = p^T W(T) p = \int_0^T \left(p^T e^{-At} B\right)\left(B^T e^{-A^T t} p\right) dt$$

Therefore

$$p^T e^{-At} B = 0, \forall t.$$

Derivating this k times and setting $t = 0$ gives $p^T A^k B = 0$.
Hence we have

$$p^T [B \;\; AB \;\; A^2 B \;\; \ldots \;\; A^{n-1} B] = 0.$$

Therefore (ii) does not hold.

$(iii) \Rightarrow (i)$ Explicit construction of $u(t)$

If $W(T)$ is invertible, then for any initial state x_0, the control signal

$$u(t) = -B^T e^{-A^T t}(W(T))^{-1} x_0$$

gives $x(T) = 0$ (check that ($\star$) some slides before is satisfied!). Hence the system is controllable.

Another interpretation of $W(T)$

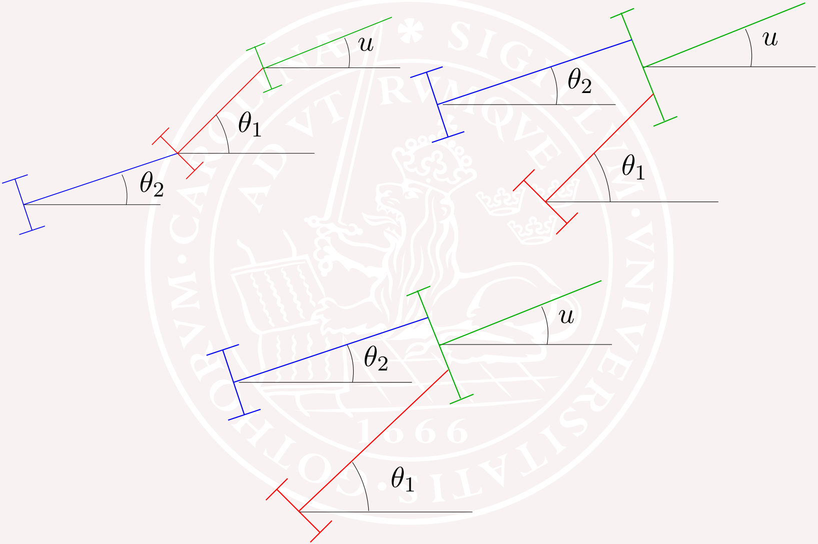
One can prove (using techniques from next lecture) that the minimal (squared) control energy, defined by $\|u\|^2 := \int_0^T |u|^2 dt$, needed to move from $x(0) = x_0$ to $x(T) = 0$ equals

$$x_0^T (W(T))^{-1} x_0$$

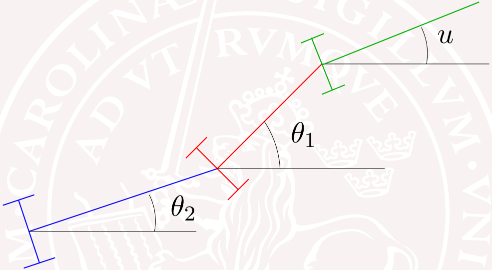
Gives nice formula for which state directions are costly to control.

$W(T)$ large in some direction means easy to control in that direction

Which trailer is controllable?

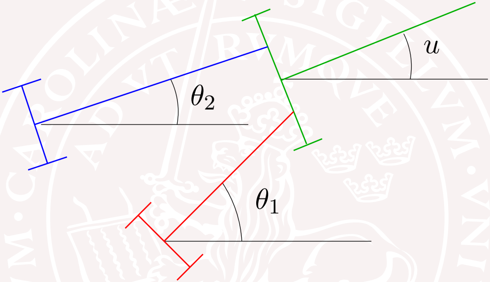


Trailer 1



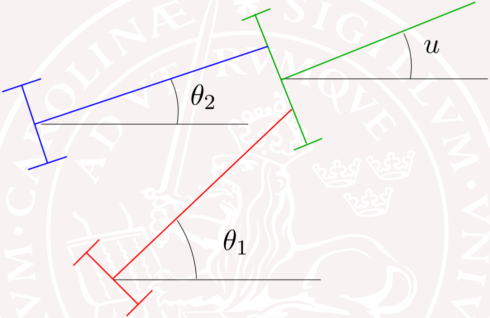
$$\begin{bmatrix}\dot\theta_1\\ \dot\theta_2\end{bmatrix} = \begin{bmatrix}-2 & 0\\ 1 & -1\end{bmatrix}\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix} + \begin{bmatrix}2\\ 0\end{bmatrix}u$$

Trailer 2



$$\begin{bmatrix}\dot\theta_1\\ \dot\theta_2\end{bmatrix} = \begin{bmatrix}-2 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix} + \begin{bmatrix}2\\ 1\end{bmatrix} u$$

Trailer 3



$$\begin{bmatrix}\dot\theta_1\\ \dot\theta_2\end{bmatrix} = \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix} + \begin{bmatrix}1\\ 1\end{bmatrix}u$$

Observability

The system

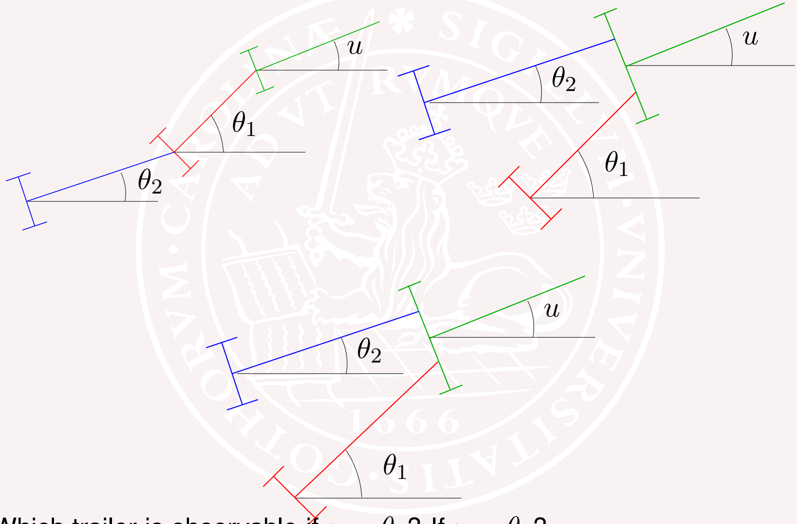
$$\begin{cases} \dfrac{dx}{dt} = Ax, \quad x(0) = x_0 \\ \quad y = Cx \end{cases}$$

is called observable if x_0 can be uniquely determined from $y_{[0,T]}$ (for any $T > 0$)

This is the same as saying that the only x_0 for which $y(t) = 0$ for all t is the trival case $x_0 = 0$

$\boxed{\text{WHY ? The audience is thinking!}}$

Which trailer is observable?



Which trailer is observable if $y = \theta_1$? If $y = \theta_2$?

Theorem - Observability Criteria

The following are equivalent

(i) The system $\dot{x} = Ax, \quad y = Cx$ is observable

(ii) rank $\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} = n$

(iii) $\widetilde{W}(T)$ is invertible for any $T > 0$

(iv) For any $\lambda \in \mathbf{C}$ we have rank $\begin{pmatrix} A - \lambda I \\ C \end{pmatrix} = n$

Here the observability Gramian $\widetilde{W}(T)$ is defined as

$$\widetilde{W}(T) = \int_0^T e^{A^T t} C^T C e^{At} dt$$

Proof that (i) $\Leftrightarrow$ (ii)

If (i) does not hold, then there is a quiet state $x_0 \neq 0$ so that

$$y(t) = Ce^{At}x_0 = 0, \quad \forall t$$

Derivating this k times and setting $t = 0$ we get $CA^k x_0 = 0$. This shows (ii) doesn't hold.

On the other hand, if (ii) does not hold then a nonzero x_0 can be found so $CA^k x_0 = 0$ for $k = 0, \ldots n-1$. By Cayley-Hamilton this means $CA^k x_0 = 0$ also for $k \geq n$, so by power expansion of e^{At}

$$y(t) = Ce^{At}x_0 = 0, \quad \forall t,$$

which says that (i) does not hold.

Proof that (ii) $\Leftrightarrow$ (iii)

This follows easily by substituting (A, B) with (A^T, C^T) in (ii) and (iii) in the controllability theorem earlier

This illustrates a so called **duality** between the two theorems

Bonus Proof of (iii)$\Rightarrow$ (i)

Maybe you didn't like the earlier proof of (ii) $\Rightarrow$ (i) that used derivation of $y(t)$. It is hard to implement in practice. If there e.g. is measurement noise on $y(t)$ we would like a better way of determining x_0.

The dual result of the construction of the energy-optimal $u(t)$ in the controllability theorem is the following calculation:

Since $y(t) = Ce^{At}x_0$ we have that

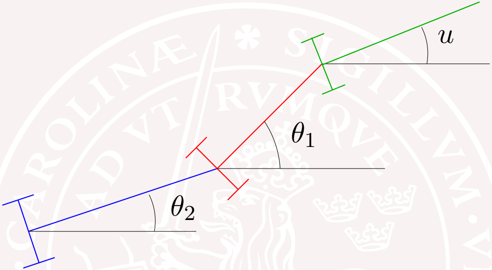
$$\int_0^T e^{A^T t} C^T y(t) dt = \widetilde{W}(T) x_0$$

When $\widetilde{W}(t)$ is invertible we can hence find x_0 by

$$x_0 = (\widetilde{W}(T))^{-1} \int_0^T e^{A^T t} C^T y(t) dt.$$

This way of determining x_0 is actually optimally robust against measurement noise, in a sense described in Lecture 6

Trailer 1

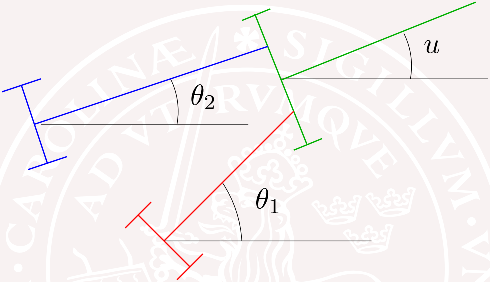


$$\begin{bmatrix}\dot\theta_1\\ \dot\theta_2\end{bmatrix} = \begin{bmatrix}-2 & 0\\ 1 & -1\end{bmatrix}\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix}$$

$$y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix}$$

$$y = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix}$$

Trailer 2



$$\begin{bmatrix}\dot\theta_1\\ \dot\theta_2\end{bmatrix} = \begin{bmatrix}-2 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix}$$

$$y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix}$$

$$y = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix}$$

Controllability – state transformation

Theorem:
If the system is noncontrollable, say rank($\mathcal{C}$) $= q < n$, then there is a state transformation $x = Vz$ so that in the new state coordinates

$$AV = V \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{pmatrix} \text{ and } B = V \begin{pmatrix} \tilde{B}_1 \\ 0 \end{pmatrix},$$

$(\tilde{A}_{11}, \tilde{B}_1)$ controllable subsystem, $q \times q$

Observability – state transformation

Theorem:
If the system is non-observable, say rank$(\mathcal{O}) = q < n$, then there is a state transformation so that in the new state coordinates

$$AV = V \begin{pmatrix} \tilde{A}_{11} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix} \text{ och } CV = \begin{pmatrix} \tilde{C}_1 & 0 \end{pmatrix},$$

$(\tilde{A}_{11}, \tilde{C}_1)$ observable subsystem, $q \times q$

Kalman's decomposition theorem

With a state transformation that splits the controllable subspace (and its complement) into nonobservable subspace and complement we get the system on a nice form

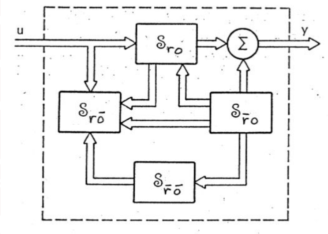
$$\frac{dx}{dt} = \begin{pmatrix} A_{11} & 0 & A_{13} & 0 \\ A_{21} & A_{22} & A_{23} & A_{24} \\ 0 & 0 & A_{33} & 0 \\ 0 & 0 & A_{43} & A_{44} \end{pmatrix} x + \begin{pmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} C_1 & 0 & C_2 & 0 \end{pmatrix} x$$

$$G(s) = C_1(sI - A_{11})^{-1}B_1$$

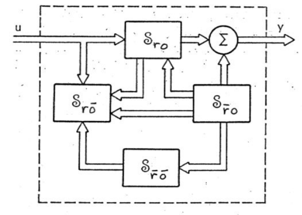
Illustrates what subparts of the system that influences the input-output behavior

Kalman's decomposition theorem

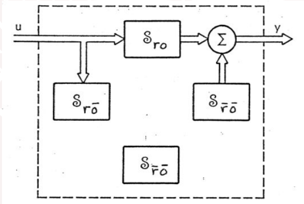


The audience if thinking: What blocks in this figure corresponds to parts 1,2,3,4 on the previous slide?

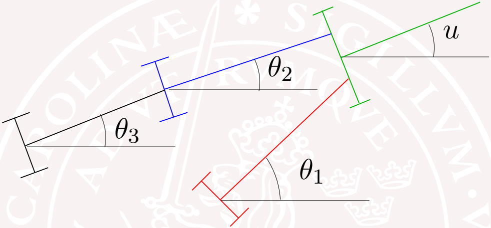
Kalman's decomposition theorem



If no common eigenvalues between any two blocks on the diagonal, then corresponding off-diagonal blocks can be eliminated by changed choice of the complementing spaces. Simplifies picture further



Trailer 4



$$\begin{bmatrix}\dot\theta_1\\\dot\theta_2\\\dot\theta_3\end{bmatrix} = \begin{bmatrix}-1 & 0 & 0\\0 & -1 & 0\\0 & 2 & -2\end{bmatrix}\begin{bmatrix}\theta_1\\\theta_2\\\theta_3\end{bmatrix} + \begin{bmatrix}1\\1\\0\end{bmatrix}u, \quad y = \begin{bmatrix}0 & 1 & 0\end{bmatrix}\begin{bmatrix}\theta_1\\\theta_2\\\theta_3\end{bmatrix}$$

What does the decomposition theorem say when $y = \theta_2$? What block is then missing?

Trailer 4 after coordinate change

$$\begin{bmatrix} \dot\theta_2 \\ \dot\theta_3 \\ \dot\theta_1 - \dot\theta_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \theta_2 \\ \theta_3 \\ \theta_1 - \theta_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta_2 \\ \theta_3 \\ \theta_1 - \theta_2 \end{bmatrix}$$

controllable and observable subsystem: θ_2

Zeros and state feedback

Remember: State-feedback does not change zeros.

Choose state feedback L that gives a pole in λ.

If the mode $x_0 e^{\lambda t}$ now becomes non-observable

$$\begin{pmatrix} A - BL - \lambda I \\ C \end{pmatrix} x_0 = 0$$

then actually λ was a zero to the system:

$$\begin{pmatrix} A - \lambda I & B \\ C & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ u_0 \end{pmatrix} = 0$$

Corresponds to cancellation of the factor $s - \lambda$ in

$$G(s) = C(sI - A + BL)^{-1} B l_r$$

Bonus: Series Connection SISO

Given two systems $n_i(s)/d_i(s) = c_i(sI - A_i)^{-1}b_i, \;\; i = 1, 2$

Then the series connection $\frac{n_2(s)}{d_2(s)}\frac{n_1(s)}{d_1(s)}$ is

- uncontrollable $\iff$ there is λ so $n_1(\lambda) = d_2(\lambda) = 0$
- unobservable $\iff$ there is z so $n_2(\lambda) = d_1(\lambda) = 0$

Proof:

Controllable, check when $\operatorname{rank} \begin{bmatrix} \lambda I - A_1 & 0 & b_1 \\ -b_2c_1 & \lambda I - A_2 & 0 \end{bmatrix} \leq n$

Observable, check when $\operatorname{rank} \begin{bmatrix} \lambda I - A_1 & 0 \\ -b_2c_1 & \lambda I - A_2 \\ 0 & c_2 \end{bmatrix} \leq n$

Cancellation in series connections

Example

$$Y(s) = \frac{s+3}{s-1} \cdot \frac{s-1}{s+2} U(s)$$

Loss of controllability of an unstable mode. Bad.

Example

$$Y(s) = \frac{s-1}{s+2} \cdot \frac{s+3}{s-1} U(s)$$

Loss of observability of an unstable mode. Also bad.

Summary

- Controllability - criteria
- Observability - criteria
- Kalman's decomposition
- Cancelled dynamics <=> lack of controllability or observability