LUND UNIVERSITY

Department of
AUTOMATIC CONTROL

Nonlinear Control and Servo Systems (FRTN05)

Exam – January 3, 2018, 08:00 – 13:00

Points and grades

All answers must include a clear motivation. The total number of points is 25. The maximum number of points is specified for each subproblem.

Preliminary grades:

3: 12 – 16.5 points

4: 17 – 21.5 points

5: 22 – 25 points

Accepted aid

All course material, except for exercises, old exams, lab instructions, and solutions of these, may be used as well as standard mathematical tables and authorized “Formelsamling i reglerteknik”/”Collection of Formulae”. Pocket calculator.

Note!

In many cases the subproblems can be solved independently of each other.

Good Luck!

1. Consider the dynamical system

$$\ddot{y} + \sin(\dot{y}) = -5y^2 + 5$$

a. Write the system on state-space form. (1 p)

b. Find the equilibrium points. (1 p)

c. Determine the character of each equilibrium point as, *e.g.*, stable node, unstable focus, *etc.* (2 p)

Solution

a. Introduce the states $x_1 = y$ and $x_2 = \dot{y}$. The state-space form is then

$$\dot{x_1} = x_2 \tag{1}$$
$$\dot{x_2} = -5x_1^2 - \sin(x_2) + 5 \tag{2}$$

b. The time derivatives are zero in the equilibria, *i.e.*,

$$0 = x_2 \tag{3}$$
$$0 = -5x_1^2 - \sin(x_2) + 5 \tag{4}$$

Insertion of the first equation into the second yields $-5x_1^2 + 5 = 0 \Leftrightarrow -x_1^2 + 1 = 0 \Leftrightarrow x_1 = \pm 1$. Hence, there are two equilibrium points, and these are $(x_1, x_2) = (-1, 0)$ and $(x_1, x_2) = (1, 0)$

c. Denote by $f(x)$ the right-hand side of state-space representation. First, we linearize the system.

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \begin{pmatrix} 0 & 1 \\ -10x_1^0 & -\cos(x_2^0) \end{pmatrix} \tag{5}$$

For (-1,0): Insertion of the equilibrium point yields

$$\frac{df}{dx} = \begin{pmatrix} 0 & 1 \\ 10 & -1 \end{pmatrix} \tag{6}$$

The eigenvalues are $\lambda_1 = 2.7016$ and $\lambda_2 = -3.7016$. Hence, this is a saddle point.

For (1,0): Insertion of the equilibrium point yields

$$\frac{df}{dx} = \begin{pmatrix} 0 & 1 \\ -10 & -1 \end{pmatrix} \tag{7}$$

The eigenvalues are $\lambda_{1,2} = -1/2 \pm \sqrt{39/4}i$. This is therefore a stable focus.

2. Consider the system given by

$$\ddot{q} = -(q^3 + \dot{q})$$

Is the system globally asymptotically stable?

Hint: It might be useful to introduce the states $x_1 = q, x_2 = \dot{q}$, and consider a function on the form $V(x) = cx_1^4 + dx_2^2$. (3 p)

Solution

Using the states as suggested by the hint, we write the system on state-space form.

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1^3 - x_2 \end{cases} \tag{8}$$

We see that the origin is an equilibrium point. With c and d positive constants, we have

$$\begin{aligned} &V(0) = 0 \\ &V(x) > 0 \quad \text{for all} \quad x \neq 0 \\ &V(x) \to \infty \quad \text{as} \quad ||x|| \to \infty \end{aligned}$$

We further have $\dot{V} = 4cx_1^3\dot{x}_1 + 2bx_2\dot{x}_2 = 4cx_1^3x_2 - 2bx_2(x_1^3 + x_2)$. Let for instance $c = 1, d = 2$. Then $\dot{V} = -4x_2^2 \leq 0$. The only solution of the state-space system that yields $\dot{V} = 0$ is $(x_1, x_2) = (0, 0)$. It therefore follows from LaSalle's theorem that the origin is globally asymptotically stable.

3. Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 + x_1(2 - x_1^2 - x_2^2) \\ \dot{x}_2 &= -x_1 + x_2(2 - x_1^2 - x_2^2) \end{aligned}$$

A solution to this system is given by

$$(x_1^0, x_2^0) = (\sqrt{2}\sin(t), \sqrt{2}\cos(t))$$

a. Linearize the system around the limit cycle. In particular, write the linearized system on the form

$$\dot{\delta} = A(t)\delta$$

where $A(t)$ is a time-dependent system matrix, and $\delta = x - x^0$. (1 p)

b. The trajectory is periodic and can thus be seen as a limit cycle. Determine whether this limit cycle is stable or not.

Hint: For this subproblem, introduce polar coordinates r and θ, *i.e.*, let

$$\begin{aligned} x_1 &= r\cos(\theta) \\ x_2 &= r\sin(\theta) \end{aligned}$$

(2 p)

Solution

a. In compact notation we have:

$$\dot{x} = f(x)$$

Introduce $\delta = x(t) - x_0(t)$ as the deviation from the nominal trajectory. We have

$$\dot{x} = \dot{x}_0 + \dot{\delta}$$

and the first order Taylor expansion of f around $x_0(t)$ is given by

$$\dot{x} = f(x_0) + \frac{\partial f(x_0)}{\partial x}\delta$$

So

$$\dot{x}_0 + \dot{\delta} = f(x_0) + \frac{\partial f(x_0)}{\partial x}\delta$$

Since $x_0(t)$ is a solution to the state equation we have $\dot{x}_0 = f(x_0)$ and thus

$$\dot{\delta} = \frac{\partial f(x_0(t))}{\partial x}\delta = A(t)\delta$$

where

$$A(t) = \begin{pmatrix} \frac{\partial f_1(x_0(t))}{\partial x_1} & \frac{\partial f_1(x_0(t))}{\partial x_2} \\ \frac{\partial f_2(x_0(t))}{\partial x_1} & \frac{\partial f_2(x_0(t))}{\partial x_2} \end{pmatrix} = \begin{pmatrix} -4\sin^2(t) & 1 - 4\sin(t)\cos(t) \\ -1 - 4\sin(t)\cos(t) & -4\cos^2(t) \end{pmatrix}.$$

b. To determine stability of the limit cycle, we introduce polar coordinates. With $r \geq 0$:

$$\begin{aligned} x_1 &= r\cos(\theta) \\ x_2 &= r\sin(\theta) \end{aligned}$$

Differentiating both sides gives

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{pmatrix} \begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix}$$

Inverting the matrix gives:

$$\begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix} = \frac{1}{r} \begin{pmatrix} r\cos(\theta) & r\sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}$$

Plugging in the state equations results in:

$$\dot{r} = r(2 - r^2) \tag{9}$$

$$\dot{\theta} = -1 \tag{10}$$

We see that the the only equilibrium points are 0 and $\sqrt{2}$ (since $r \geq 0$). Linearizing around $r = \sqrt{2}$ (i.e. the limit cycle) gives:

$$\dot{\tilde{r}} = -4\tilde{r}$$

Since the eigenvalue is strictly negative, this implies that $r = \sqrt{2}$ is a locally asymptotically stable equilibrium point. Hence the limit cycle is stable.

4. Consider an asymptotically stable linear system $G(s)$, of which the Nyquist plot is shown in Figure 1. The linear system is connected with negative feedback to a static nonlinearity, Ψ, as shown in Figure 2.

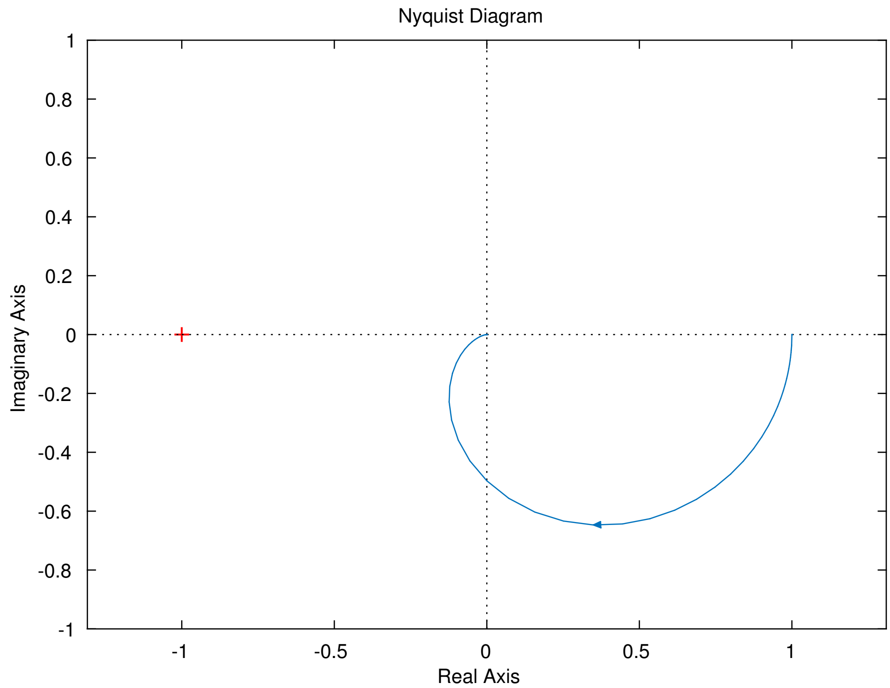

Figure 1 Nyquist plot of $G(s)$ in Problem 4.

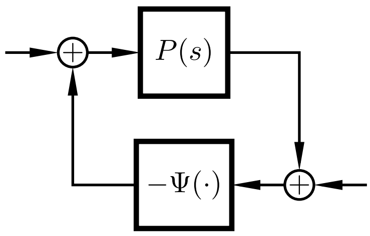

Figure 2 Feedback in Problem 4.

a. Use the Circle criterion to determine a sector for Ψ, as large as you can, such that the feedback system is BIBO stable. (2 p)

b. Is the system BIBO stable for $\Psi(\cdot) = \sin(\cdot)$? (1 p)

Solution

a. Figure 3 shows that the Nyquist plot lies inside the disk $D(-0.5; 1)$. The maximal stability sector is therefore $(-1; 2)$.

b. The function $\Psi(x) = \sin(x)$ is for instance bounded by the lines $l_1 = -x$ and $l_2 = x$. Since this is within the stability sector $(-1; 2)$, the feedback system is stable.

5. A team of engineers are working together to build a self-driving electric car. They have a sketch ready, but still a lot of work remains. One subproblem consists of designing a sliding mode controller for the system

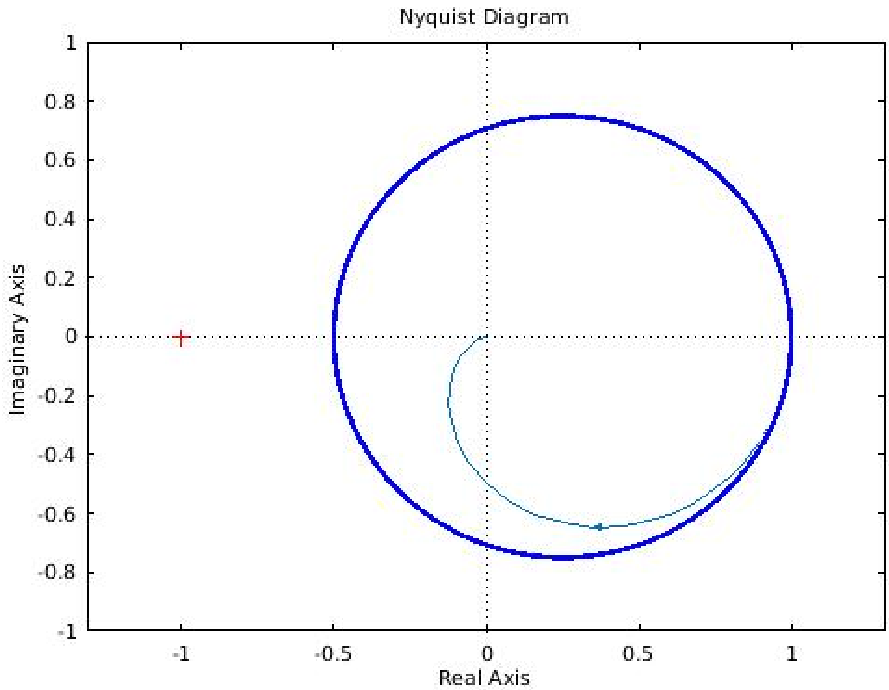

Figure 3 Nyquist plot of $G(s)$

$$\dot{x}_1 = x_2 + u$$
$$\dot{x}_2 = x_1$$

with the switch function $\sigma(x) = x_1 + 2x_2$. They turn to you for advice, since they lack experience in control design.

a. Design a sliding mode controller with the switch function $\sigma(x)$. (2 p)

b. Determine the sliding set and the sliding dynamics. (1 p)

Solution

a. The control law is

$$u = -\frac{p^T Ax}{p^T B} - \frac{\mu}{p^T B}\text{sign}(\sigma(x)) \tag{11}$$

where

$$\sigma(x) = p^T x = 0 \tag{12}$$

Hence, we have $p^T = [1\ 2]$, and this yields that

$$u = -(2x_1 + x_2) - \mu\text{sign}(x_1 + 2x_2) \tag{13}$$

b. The sliding set is where $\sigma(x) = x_1 + 2x_2 = 0$, *i.e.*, the line $x_1 + 2x_2 = 0$.

To find the sliding dynamics, we first determine the equivalent control u_{eq}. On the sliding set we have

$$\dot{\sigma} = \dot{x}_1 + 2\dot{x}_2 = x_2 + u_{\text{eq}} + 2x_1 = 0$$

and therefore

$$u_{\text{eq}} = -2x_1 - x_2$$

The sliding dynamics is

$$\begin{aligned} \dot{x}_1 &= x_2 + u_{\text{eq}} = -2x_1 \\ \dot{x}_2 &= x_1 \end{aligned}$$

6. Consider the system

$$\frac{\mathrm{d}^3 z}{\mathrm{d}t^3} + \frac{\mathrm{d}^2 z}{\mathrm{d}t^2} + \frac{\mathrm{d}z}{\mathrm{d}t} = -\frac{1}{3}z^3$$

a. This can be seen as a negative feedback connection between a linear system $P(s)$ and a static nonlinearity $f(x) = \frac{1}{3}x^3$. Determine $P(s)$. (2 p)

b. Calculate the describing function of the nonlinearity $f(x) = \frac{1}{3}x^3$. (2 p)

(*Hint:* $\int_0^{2\pi} \sin(x)^4 \mathrm{d}x = \frac{3\pi}{4}$)

c. Analyze the existence, amplitude and frequency of a possible limit cycle. (2 p)

Solution

a. The Laplace transform between $-f$ and z results in

$$P(s) = \frac{1}{s(s^2 + s + 1)}.$$

b. The function is odd, which implies that it is real.

$$b_1 = \frac{A^3}{3\pi} \int_0^{2\pi} \sin(\phi)^4 \mathrm{d}\phi = \frac{A^3}{4},$$

which gives that the describing function

$$N(A) = \frac{A^2}{4}.$$

c. We want to find out the points where $\text{Im}P(i\omega) = 0$. Some calculations gives that

$$\text{Im}P(i\omega) = \frac{-(1-\omega^2)}{\omega((1-\omega^2)^2+\omega^2)},$$

which in its turn gives that $\omega = 1$. Finally, this yields that

$$P(i) = -1 = -\frac{1}{N(A)} = -\frac{4}{A^2} \Rightarrow A = 2.$$

To conclude: The frequency of the limit cycle is $\omega = 1$ rad/s and its amplitude is $A = 2$.

7. Isabelle and Antonio work as control engineers in a toy factory. This year, they aim to improve the number of toys produced per year, and have encountered the following optimal control problem as a main challenge.

$$\text{Minimize } \int_0^1 4u(t)^2\, \mathrm{d}t + 6x(1)^2$$

subject to

$$\begin{aligned} \dot{x} &= u \\ x(0) &= 1 \end{aligned}$$

Unfortunately, they are not very experienced in optimal control, and the rest of the staff are on Winter vacation. They kindly ask you for help, since they know that you have studied nonlinear control. Derive the optimal control law for the problem above.

(3 p)

Solution

We have a fixed final time, $t_f = 1$. Further,
$\Phi(x(t_f)) = 6x(t_f)^2$
$L = 4u^2$
$f(x,u) = u$
Hamiltonian:

$$H = L + \lambda f = 4u^2 + \lambda u$$

Adjoint equation:

$$\begin{aligned} \dot{\lambda} &= -H_x = 0 \\ \lambda(t_f) &= \Phi_x(x(t_f)) = 12x(t_f) \end{aligned}$$

Therefore, it can be noted that λ is constant, $\lambda(t) = 12x(t_f)$.
Optimality conditions:
Minimizing H with respect to u gives

$$H_u = 8u + \lambda = 0 \Leftrightarrow u = -\frac{\lambda}{8}$$

Hence, $u = -\frac{3}{2}x(t_f)$.
The last step is to insert this into the system equation, which yields

$$\dot{x} = -\frac{3}{2}x(t_f) \Leftrightarrow x(t_f) - x(0) = \int_0^{t_f} -\frac{3}{2}x(t_f)\,dt \Leftrightarrow x(1) - 1 = -\frac{3}{2}x(1) \Leftrightarrow x(1) = \frac{2}{5}$$

Hence, the optimal control law is $u = -\frac{3}{2}x(t_f) = -\frac{3}{5}$