## Lecture 6 — Describing function analysis

**Today's Goal:** *To be able to*

- *Derive describing functions for static nonlinearities*
- *Predict stability and existence of periodic solutions through describing function analysis*

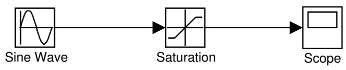
**Material:**

- Chapter 14 in Glad & Ljung
- Chapter 7.2 (pp.280–290) in Khalil
- Lecture notes

## Course Outline

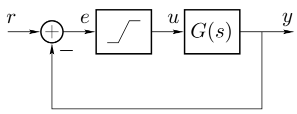
| | |
|---|---|
| Lecture 1-3 | Modelling and basic phenomena (linearization, phase plane, limit cycles) |
| Lecture 2-6 | Analysis methods (Lyapunov, circle criterion, describing functions) |
| Lecture 7-8 | Common nonlinearities (Saturation, friction, backlash, quantization) |
| Lecture 9-13 | Design methods (Lyapunov methods, Sliding mode & optimal control) |
| Lecture 14 | Summary |

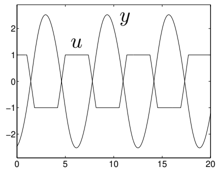
## Example: saturated sinusoidals




The "effective gain" ( the ratio $\dfrac{\text{sat}(A\sin\omega t)}{A\sin\omega t}$) varies with the input signal amplitude $A$.

## Motivating Example




$G(s) = \dfrac{4}{s(s+1)^2}$ and $u = \text{sat}\,(e)$ gives stable oscillation for $r = 0$.

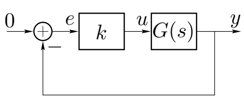
- How can the oscillation be predicted?

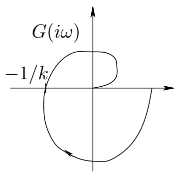
Q: What is the amplitude/topvalue of $u$ and $y$? What is the frequency?

## Recall the Nyquist Theorem

Assume $G(s)$ stable. For what frequency $\omega$ and gain $k$ is

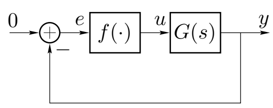
$$k \cdot G(i\omega) = -1?$$

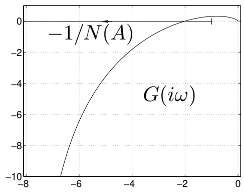



## Motivating Example (cont'd)

Heuristic reasoning:




Let $N(A)$ be a complex number such that
$f(A\sin\omega t) \approx |N(A)|A\sin[\omega t + \arg N(A)]$.

Heuristic reasoning: For what frequency and what amplitude is "the loop gain" $N(A)\cdot G(iw) = -1$?

The intersection of the $-1/N(A)$ and the Nyquist curve $G(i\omega)$ predicts amplitude and frequency.

- How do we derive the **describing function** $N(A)$?
- Does the intersection predict a stable oscillation?
- Are the estimated amplitude and frequency accurate?

## Fourier Series

Every periodic function $u(t) = u(t+T)$ has a Fourier series expansion

$$\begin{aligned} u(t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos n\omega t + b_n \sin n\omega t) \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty}\sqrt{a_n^2+b_n^2}\,\sin[n\omega t + \arctan(a_n/b_n)] \end{aligned}$$

where $\omega = 2\pi/T$ and

$$a_n = \frac{2}{T}\int_0^T u(t)\cos n\omega t\,dt \qquad b_n = \frac{2}{T}\int_0^T u(t)\sin n\omega t\,dt$$

**Note:** Sometimes we make the change of variable $t \to \phi/\omega$

## The Fourier Coefficients are Optimal

The finite expansion

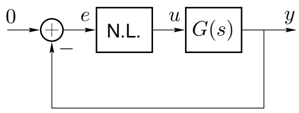
$$\widehat{u}_k(t) = \frac{a_0}{2} + \sum_{n=1}^{k} (a_n \cos n\omega t + b_n \sin n\omega t)$$

solves

$$\min_{\{\hat{a}_n, \hat{b}_n\}_{1 \le n \le k}} \frac{2}{T} \int_0^T \left[u(t) - \widehat{u}_k(t)\right]^2 dt \qquad T = \frac{2\pi}{\omega}$$

if $\{a_n,\, b_n\}$ are the Fourier coefficients.

## The Key Idea




Assume $e(t) = A \sin \omega t$ and $u(t)$ periodic. Then

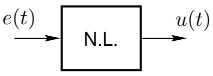
$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2} \sin[n\omega t + \arctan(a_n/b_n)]$$

If $|G(in\omega)| \ll |G(i\omega)|$ for $n = 2, 3, \ldots$ and $a_0 = 0$, then

$$y(t) \approx |G(i\omega)| \sqrt{a_1^2 + b_1^2} \sin[\omega t + \arctan(a_1/b_1) + \arg G(i\omega)]$$

Find periodic solution by matching coefficients in $y = -e$.

## Definition of Describing Function

The **describing function** of  is


$$N(A, \omega) := \frac{b_1(\omega) + i a_1(\omega)}{A}$$

$$a_1(\omega) := \frac{\omega}{\pi} \int_0^{2\pi/\omega} u(t) \cos(\omega t) dt \qquad b_1(\omega) := \frac{\omega}{\pi} \int_0^{2\pi/\omega} u(t) \sin(\omega t) dt$$

where $u(t)$ is the output corresponding to $e(t) := A \sin(\omega t)$

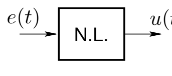
If $G$ is low pass and $a_0 = 0$, then

$$\widehat{u}_1(t) = |N(A, \omega)| A \sin[\omega t + \arg N(A, \omega)]$$

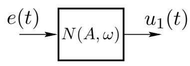
can be used instead of $u(t)$ to analyze the system.

Amplitude dependent gain and phase shift!

$$e(t) = A \sin \omega t = \operatorname{Im}\, (A e^{i\omega t})$$




$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t)$$

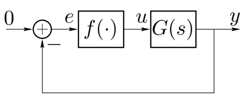


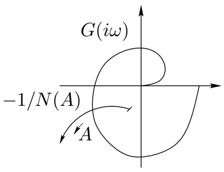

$$u_1(t) = a_1 \cos(\omega t) + b_1 \sin(\omega t)$$
$$= \operatorname{Im}\, (N(A, \omega) A e^{i\omega t})$$

where the **describing function** is defined as

$$N(A, \omega) = \frac{b_1(\omega) + i a_1(\omega)}{A} \Longrightarrow U(i\omega) \approx N(A, \omega) E(i\omega)$$

## Existence of Limit Cycles

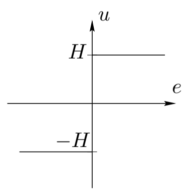


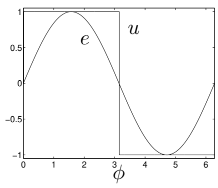




$$y = G(i\omega) u \approx -G(i\omega) N(A) y \quad \Rightarrow \quad G(i\omega) = -\frac{1}{N(A)}$$

The intersections of $G(i\omega)$ and $-1/N(A)$ give $\omega$ and $A$ for possible limit cycles.

## Describing Function for a Relay

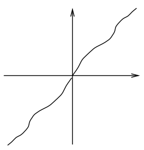



$$a_1 = \frac{1}{\pi} \int_0^{2\pi} u(\phi) \cos \phi \, d\phi = 0$$
$$b_1 = \frac{1}{\pi} \int_0^{2\pi} u(\phi) \sin \phi \, d\phi = \frac{2}{\pi} \int_0^{\pi} H \sin \phi \, d\phi = \frac{4H}{\pi}$$

The describing function for a relay is thus $N(A) = \dfrac{4H}{\pi A}$.

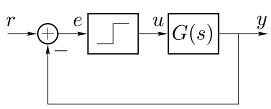
## Describing Function for Odd Static Nonlinearities

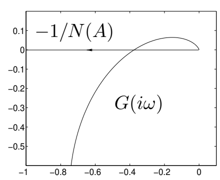


Assume $f(\cdot)$ and $g(\cdot)$ are odd static nonlinearities (i.e., $f(-e) = -f(e)$) with describing functions $N_f$ and $N_g$. Then,

- $\operatorname{Im} N_f(A, \omega) = 0$, coeff. $(a_1 \equiv 0)$
- $N_f(A, \omega) = N_f(A)$
- $N_{\alpha f}(A) = \alpha N_f(A)$
- $N_{f+g}(A) = N_f(A) + N_g(A)$

## Limit Cycle in Relay Feedback System



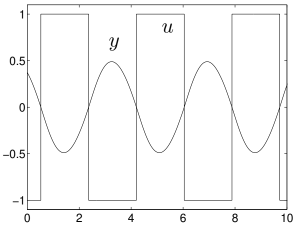

$$G(s) = \frac{3}{(s+1)^3} \quad \text{with feedback} \quad u = -\operatorname{sgn} y$$

$$-3/8 = -1/N(A) = -\pi A/4 \quad \Rightarrow \quad A = 12/8\pi \approx 0.48$$

$$G(i\omega) = -3/8 \quad \Rightarrow \quad \omega = \sqrt{3} \approx 1.7, \;\; T = 2\pi/\omega \approx 3.6$$

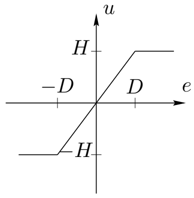
## Limit Cycle in Relay Feedback System (cont'd)

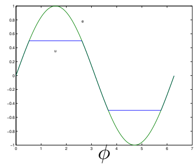
The prediction via the describing function agrees very well with the true oscillations:




$G$ filters out almost all higher-order harmonics.

## Describing Function for a Saturation




Let $e(t) = A\sin\omega t = A\sin\phi$. First set $H = D$. If $A \le D$ then $N(A) = 1$, if $A > D$ then for $\phi \in (0, \pi)$

$$u(\phi) = \begin{cases} A\sin\phi, & \phi \in (0, \phi_0) \cup (\pi - \phi_0, \pi) \\ D, & \phi \in (\phi_0, \pi - \phi_0) \end{cases}$$

where $\phi_0 = \arcsin D/A$. Moreover, $u(\phi) = -u(2\pi - \phi)$.

## Describing Function for a Saturation (cont'd)

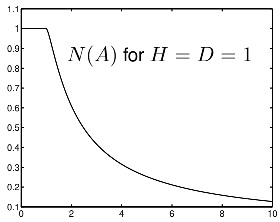
$$\begin{aligned} a_1 &= \frac{1}{\pi}\int_0^{2\pi} u(\phi)\cos\phi\, d\phi \\ &= \frac{1}{\pi}\int_0^{\pi} u(\phi)\cos\phi\, d\phi + \frac{1}{\pi}\int_0^{\pi} u(2\pi - \phi)\cos\phi\, d\phi = 0 \end{aligned}$$

$$\begin{aligned} b_1 &= \frac{1}{\pi}\int_0^{2\pi} u(\phi)\sin\phi\, d\phi = \frac{4}{\pi}\int_0^{\pi/2} u(\phi)\sin\phi\, d\phi \\ &= \frac{4A}{\pi}\int_0^{\phi_0} \sin^2\phi\, d\phi + \frac{4D}{\pi}\int_{\phi_0}^{\pi/2} \sin\phi\, d\phi \\ &= \frac{A}{\pi}\Big(2\phi_0 + 2\sin\phi_0\cos\phi_0\Big) \end{aligned}$$

## Describing Function for a Saturation (cont'd)

If $H = D$ then

$$\begin{aligned} N(A) &= \frac{1}{\pi}\Big(2\phi_0 + 2\sin\phi_0\cos\phi_0\Big), \qquad A \ge D \\ &= \frac{2}{\pi}\left(\arcsin\frac{D}{A} + \frac{D}{A}\sqrt{1 - \frac{D^2}{A^2}}\right) \end{aligned}$$




NOTE: dependance of A shows up in $\phi_0 = \arcsin D/A$

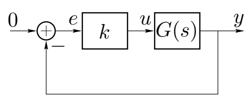
## 3 minute exercise:

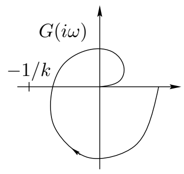
*What oscillation amplitude and frequency do the describing function analysis predict for the "Motivating Example"?*

## The Nyquist Theorem

Assume $G(s)$ stable, and $k$ is positive gain.

- The closed-loop system is unstable if the point $-1/k$ is encircled by $G(i\omega)$
- The closed-loop system is stable if the point $-1/k$ is not encircled by $G(i\omega)$

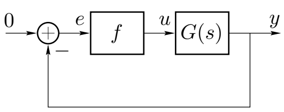


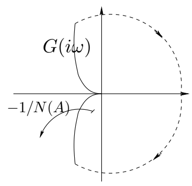




## How to Predict Stability of Limit Cycles

Assume $G(s)$ stable. For a given $A = A_0$:

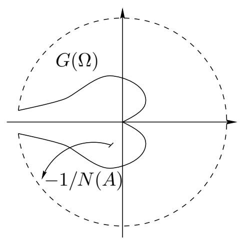
- $A$ increases if the point $-1/N_f(A_0)$ is encircled by $G(i\omega)$
- $A$ decreases otherwise




A stable limit cycle is predicted

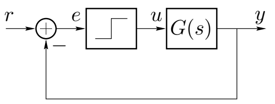
## How to Predict Stability of Limit Cycles

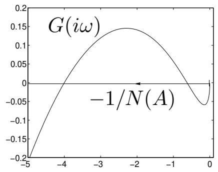



An unstable limit cycle is predicted

An intersection with amplitude $A_0$ is unstable if $A < A_0$ gives decreasing amplitude and $A > A_0$ gives increasing.

## Stable Periodic Solution in Relay System



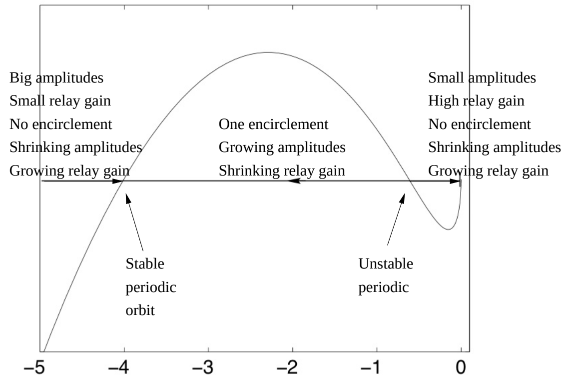




$$G(s) = \frac{(s+10)^2}{(s+1)^3} \quad \text{with feedback} \quad u = -\text{sgn}\, y$$

gives one stable and one unstable limit cycle. The left most intersection corresponds to the stable one.

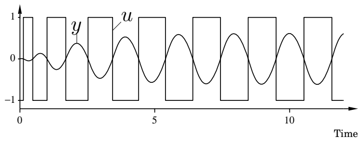
## Periodic Solutions in Relay System

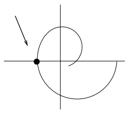
The relay gain $N(A)$ is higher for small $A$:

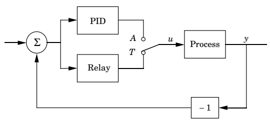



## Automatic Tuning of PID Controller

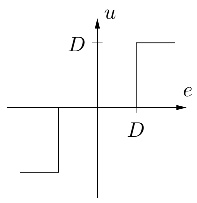
Period and amplitude of relay feedback limit cycle can be used for autotuning.

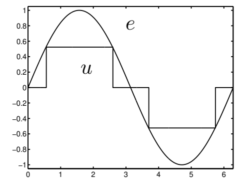





## Describing Function for a dead-zone relay




Let $e(t) = A \sin \omega t = A \sin \phi$. Then for $\phi \in (0, \pi)$

$$u(\phi) = \begin{cases} 0, & \phi \in (0, \phi_0) \\ D, & \phi \in (\phi_0, \pi - \phi_0) \end{cases}$$

where $\phi_0 = \arcsin D/A$ (if $A \geq D$)

## Describing Function for a dead-zone relay–cont'd.

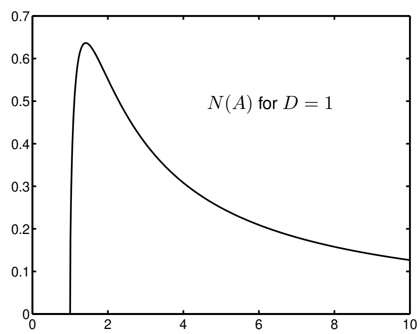
$$a_1 = \frac{1}{\pi} \int_0^{2\pi} u(\phi) \cos \phi \, d\phi = 0$$

$$b_1 = \frac{1}{\pi} \int_0^{2\pi} u(\phi) \sin \phi d\phi = \frac{4}{\pi} \int_{\phi_0}^{\pi/2} D \sin \phi d\phi$$

$$= \frac{4D}{\pi} \cos \phi_0 = \frac{4D}{\pi} \sqrt{1 - D^2/A^2}$$

$$N(A) = \begin{cases} 0, & A < D \\ \dfrac{4}{\pi A} \sqrt{1 - D^2/A^2}, & A \geq D \end{cases}$$

## Plot of Describing Function for dead-zone relay




Notice that $N(A) \approx 1.3/A$ for large amplitudes

## Pitfalls

Describing function analysis can give erroneous results.

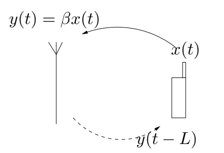
- DF analysis may predict a limit cycle, even if it does not exist.
- A limit cycle may exist, even if DF analysis does not predict it.
- The predicted amplitude and frequency are only approximations and can be far from the true values.

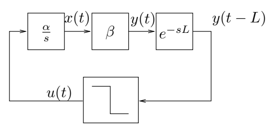
## Example

The control of output power $x(t)$ from a mobile telephone is critical for good performance. One does not want to use too large power since other channels are affected and the battery length is decreased. Information about received power is sent back to the transmitter and is used for power control. A very simple scheme is given by

$$\begin{aligned} \dot{x}(t) &= \alpha u(t) \\ u(t) &= -\text{sign}\, y(t - L), \qquad \alpha, \beta > 0 \\ y(t) &= \beta x(t). \end{aligned}$$

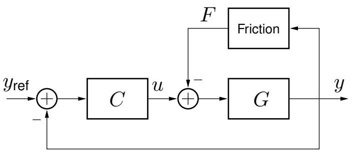
Use describing function analysis to predict possible limit cycles.



## Accuracy of Describing Function Analysis

Control loop with friction $F = \text{sgn}\, y$:

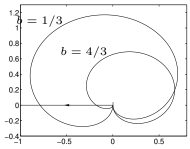



Corresponds to

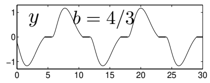
$$\frac{G}{1+GC} = \frac{s(s-b)}{s^3+2s^2+2s+1} \quad \text{with feedback} \quad u = -\text{sgn}\, y$$

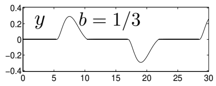
The oscillation depends on the zero at $s = b$.

## Accuracy of Describing Function Analysis




DF predicts period times and ampl. $(T, A)_{b=4/3} = (11.4, 1.00)$ and $(T, A)_{b=1/3}$ =(17.3,0.23)




Simulation:

$$(T, A)_{b=4/3} = (12, 1.1)$$

$$(T, A)_{b=1/3} = (22, 0.28)$$

Accurate results only if $y$ is sinusoidal!

## Analysis of Oscillations—A summary

There exist both time-domain and frequency-domain methods to analyze oscillations.

Time-domain:

- Poincaré maps and Lyapunov functions
- Rigorous results but hard to use for large problems

Frequency-domain:

- Describing function analysis
- Approximate results
- Powerful graphical methods

## Today's Goal

*To be able to*

- *Derive describing functions for static nonlinearities*
- *Predict stability and existence of periodic solutions through describing function analysis*

## Next Lecture

- Saturation and antiwindup compensation
- Lyapunov analysis of phase locked loops
- Friction compensation