LUND UNIVERSITY

Department of
AUTOMATIC CONTROL

Nonlinear Control and Servo Systems (FRTN05)

Exam - April 24, 2019, 8 am – 13 pm

Points and grades

All answers must include a clear motivation. The total number of points is 25. The maximum number of points is specified for each problem.

Preliminary grades:

3: 12 – 16.5 points

4: 17 – 21.5 points

5: 22 – 25 points

Accepted aid

All course material, except for exercises, old exams, and solutions of these, may be used as well as standard mathematical tables and authorized "Formelsamling i reglerteknik"/"Collection of Formulae". Pocket calculator.

Note!

In many cases the sub-problems can be solved independently of each other.

Good Luck!

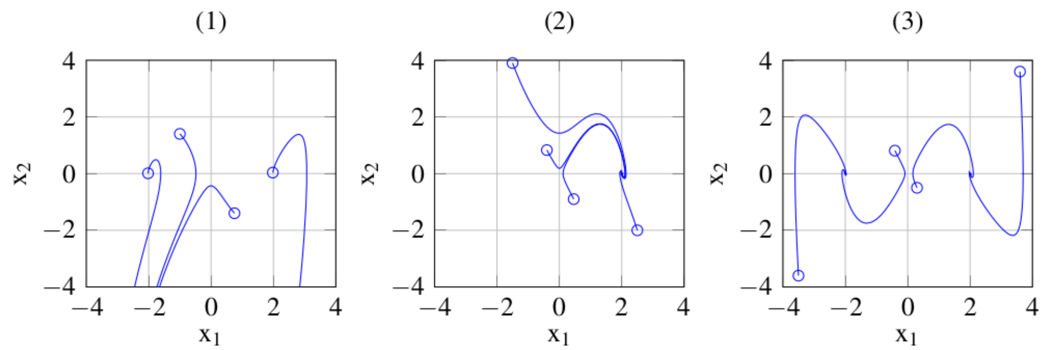

Figure 1 The simulated systems in problem 1. The initial value for each simulated trajectory is marked with a circle.

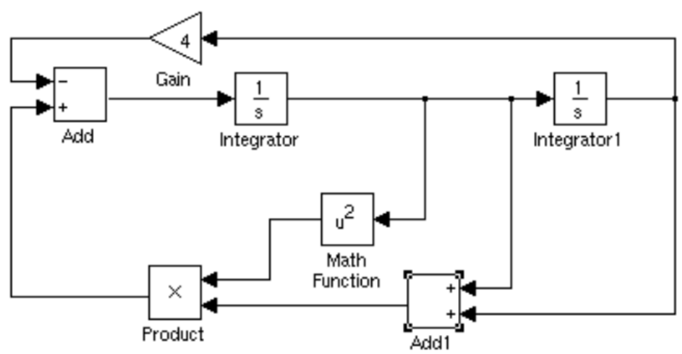

Figure 2 An (incorrect) attempt to describe the system in problem 1 in Simulink.

1. Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1^2 x_2 - x_1^3 + 4x_1 \end{aligned} \tag{1}$$

a. Determine all equilibria of the system. (1 p)

b. Classify all such equilibria according to the local behavior of the system in their proximity. (2 p)

c. Each of the three plots in Figure 1 depicts the trajectories of some dynamical system. The initial value of each simulated trajectory is marked with a small circle. Which one(s) of the three plots is compatible with the dynamical system (1)? Motivate your answer. (1 p)

d. An incorrect attempt to simulate system (1) in Simulink is displayed in Figure 2. Please sketch a correct implementation of (1). (1 p)

Solution

Solution:

a. The equilibria are found by setting all derivatives to zero:

$$\begin{cases} 0 &= x_2 \\ 0 &= -x_1^2 x_2 - x_1^3 + 4x_1 \end{cases} \Leftrightarrow \begin{cases} x_2 &= 0 \\ 0 &= x_1(x_1^2 - 4) \end{cases}$$

and the solutions to this equation system are the three equilibrium points (-2,0), (0,0), and (2,0).

b. The local behavior in the proximity of the equilibria is found by investigating the linearizations around these points. The A-matrix will be given as

$$A = \frac{\partial f}{\partial x}\bigg|_{x=x_0} = \left[\begin{array}{cc} 0 & 1 \\ -2x_1x_2 - 3x_1^2 + 4 & -x_1^2 \end{array}\right]\Bigg|_{x=x_0}$$

The equilibrium in (0,0) gives

$$A = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix}$$

which has eigenvalues in 2 and -2 and the equilibrium is hence a saddle point.

The equilibria in ($\pm$2,0) gives

$$A = \begin{bmatrix} 0 & 1 \\ -8 & -4 \end{bmatrix}$$

which has eigenvalues in $-2 \pm 2i$ and the equilibria are hence stable foci.

c. For the given system, the equilibrium point in (0,0) is a saddle point, and the other two are stable foci. Subplot (1) is clearly not from the considered system, as initial values in the equilibrium points (-2,0) and (2,0) diverges.

In (2) there seems to be a saddle point in the origin and a stable focus in (2,0). This corresponds well with the properties of the given system (what happens around (-2,0) is not clear from the given simulated system trajectories).

In (3) there seems to be stable focuses in (-2,0) and (2,0), and a saddle point in the origin. This corresponds well with the given system.

To summarize, (2) and (3) may have been made with the given system in the problem.

d. Two errors are made in the implementation.

1. The addition block named "Add" should switch the signs for the inputs
2. The input to the quadratic function block should come from the output of the second integrator ("Integrator1") instead of the first integrator.

A correct implementation is displayed in Figure 3.

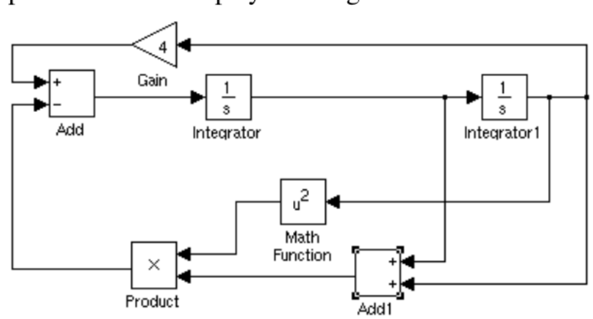

Figure 3 A correct implementation of the system in problem 1 in Simulink.

2. A simple model of a laser is given by

$$\frac{dP(t)}{dt} = KN(t)P(t) - \gamma_c P(t) \tag{2a}$$

$$\frac{dN(t)}{dt} = J - \frac{1}{T_1}N(t) - KN(t)P(t) \tag{2b}$$

where P is the number of photons in the laser cavity, N is the number of carriers, K is a gain constant, γ_c is the decay rate of cavity photons, J is the carrier pump rate, and T_1 is the carrier life time.

a. Lasers only lase when the pump rate J is larger than some threshold value J_{thr}. Lasing corresponds to that the equations (2) have an equilibrium (P_0, N_0) with $P_0 > 0, N_0 > 0$.

For which pump rates J does the laser lase? I.e., find the lasing threshold J_{thr}.
Note: The parameters γ_c, K and T_1 are all positive. (1 p)

b. Consider the following, typical, normalized parameter values, $\gamma_c = K = 1$, $J = 0.002$, $T_1 = 1000$. For these values it holds that $J > J_{\text{thr}}$, so the equations (2) have an equilibrium point $P_0 > 0$, $N_0 > 0$. How would you expect the variable $P(t)$ to evolve (as a function of t) for initial conditions close to the equilibrium point? Make a rough sketch and motivate your answer. (1 p)

Solution

a. An equilibrium point N_0, P_0 satisfies

$$0 = KN_0P_0 - \gamma_c P_0$$
$$0 = J - N_0/T_1 - KN_0P_0$$

From the first equation we get two solutions for P_0. One possibility is $P_0 = 0$ which we can discard since we are looking for an equilibrium point with $P_0 > 0$, $N_0 > 0$. The other possibility is

$$N_0 = \gamma_c/K > 0. \tag{3}$$

By solving the second equation for P_0 and using that $N_0 = \gamma_c/K$, we get

$$P_0 = \frac{J - N_0/T_1}{KN_0} = \frac{J - \gamma_c/(KT_1)}{\gamma_c}. \tag{4}$$

Thus we have that P_0 is positive if $J > \gamma_c/(KT_1) = J_{\text{thr}}$.

b. Linearizing the system around the equilibrium point given by (3), (4) we get the following system matrix

$$A = \begin{bmatrix} KN - \gamma_c & KP \\ -KN & -KP - 1/T_1 \end{bmatrix}\Bigg|_{P=P_0, N=N_0}$$

For the given parameter values we have $N_0 = 1$, $P_0 = 0.001$, and the system matrix for the linearized system becomes

$$A = \begin{bmatrix} 0 & 0.001 \\ -1 & -0.002 \end{bmatrix}.$$

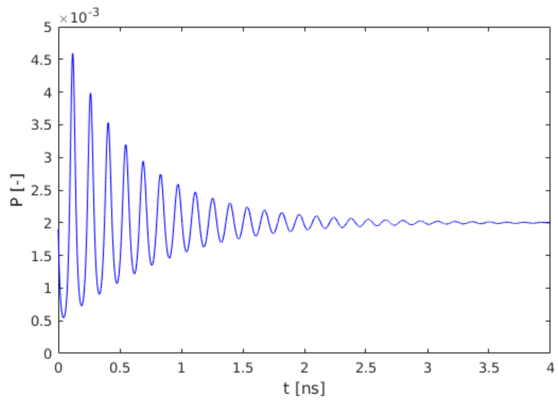

Figure 3 Relaxation oscillations from initial conditions $P(0) = 0.95P_0$, $N(0) = 0.95N_0$
Remark: The normalization of the parameters correspond to a time scale of pico seconds.

The eigenvalues of A are $\lambda_{1,2} = -0.001 \pm \sqrt{0.001^2 - 0.001} \approx -0.001 \pm 0.032i$ which corresponds to a stable focus.

c. Based on that the equilibrium point of the system is a stable focus, we expect the transient response to oscillate and converge to P_0, see Figure 3. These oscillations are called *relaxation oscillations* in the laser literature.

3. The Euler equations for a rotating rigid spacecraft are given by

$$\begin{aligned} J_1\dot{\omega}_1 &= (J_2 - J_3)\omega_2\omega_3 + u_1 \\ J_2\dot{\omega}_2 &= (J_3 - J_1)\omega_3\omega_1 + u_2 \\ J_3\dot{\omega}_3 &= (J_1 - J_2)\omega_1\omega_2 + u_3 \end{aligned}$$

where ω_i are the components of the angular velocity vector ω along the principal axes, J_i are the corresponding moment of inertia, and u_i are control torques applied along the principal axes.

a. Assume $u_1 = u_2 = u_3 = 0$, show that the origin $\omega_1 = \omega_2 = \omega_3 = 0$ is stable. Is it asymptotically stable?

Hint: Use a quadratic Lyapunov Function candidate of the form $a\omega_1^2 + b\omega_2^2 + c\omega_3^2$.
(2.5 p)

b. Suppose that toque feedback is applied according to $u_i = -k_i\omega_i$, where all $k_i > 0$. Show that the origin is globally asymptotically stable. (1 p)

Solution

a. We get

$$\dot{V} = 2\left(a\omega_1\frac{J_2 - J_3}{J_1}\omega_2\omega_3 + b\omega_2\frac{J_3 - J_1}{J_2}\omega_1\omega_3 + c\omega_3\frac{J_1 - J_2}{J_3}\omega_1\omega_2\right)$$

letting $a = J_1, b = J_2, c = J_3$ gives

$$\dot{V} = 2\omega_1\omega_2\omega_3\left((J_2 - J_3) + (J_3 - J_1) + (J_1 - J_2)\right) = 0, \quad \forall\omega$$

since V is a Lyapunov Function with $V(0) = 0$ and $V(\omega) > 0$, $\forall \omega \neq 0$, we get that $\omega = 0$ is stable.

Since $dV/dt = 0$ the trajectories will stay on whatever level surface of V that they started on. Thus the trajectories will not go to zero for non-zero initial conditions.

b. Using the same values for a, b, c we get

$$\begin{aligned}\dot{V} &= 2\left(\omega_1(J_2 - J_3)\omega_2\omega_3 - k_1\omega_1^2 + \omega_2(J_3 - J_1)\omega_1\omega_3 - k_2\omega_2^2 + \omega_3(J_1 - J_2)\omega_1\omega_2 - \omega_3^2\right) \\ &= -2(k_1\omega_1^2 + k_2\omega_2^2 + k_3\omega_3^2) < 0, \quad \forall \omega \neq 0\end{aligned}$$

so the origin is globally asymptotically stable since the requirements for a Lyapunov function is satisfied and $V(\omega) \to \infty$ when $\|\omega\| \to \infty$.

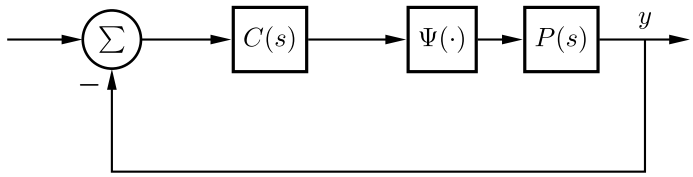

Figure 4 Feedback interconnection of a double-tank process with nonlinear pump dynamics, and a controller $C(s)$.

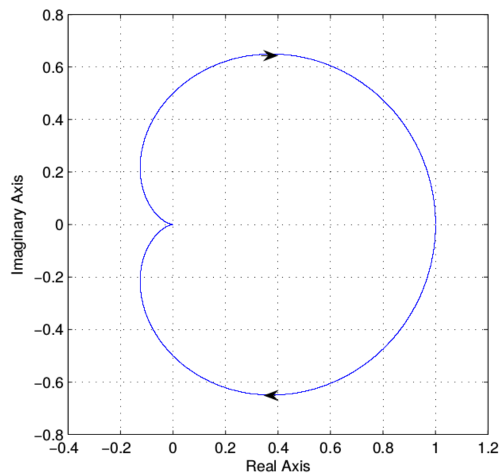

Figure 5 Nyquist curve of a double water tank.

4. A double tank process, similar to the one from the laboratory exercises in the basic course, can be modeled as

$$P(s) = \frac{1}{(s+1)^2}$$

This system has the Nyquist curve, $P(i\omega)$, in Figure 5. The water level in the lower tank is controlled by a proportional controller $C(s) = K_p$. The pump which pumps water into the upper tank is nonlinear, and is described by the static nonlinearity $\Psi(\cdot)$. The whole feedback connection is seen in Figure 4.

a. Is the feedback system, in Figure 4, BIBO stable if $K_p = 1$ and the pump nonlinearity Ψ belongs to the sector $(\alpha, \beta) = (0.5, 5)$? Motivate your answer! (1 p)

b. Determine an upper bound on K_p , such that the feedback connection will be BIBO stable when the nonlinearity belongs to a sector $(\alpha, \beta) = (-0.5, 0.5)$. Calculations are needed in your motivation! (1 p)

c. Determine an upper bound on K_p such that the feedback connection is be BIBO stable when the nonlinearity belongs to a sector $(\alpha, \beta) = (0, 10)$. Calculations are needed in your motivation! (1 p)

Solution

We can use the circle criterion to solve all three subproblems.

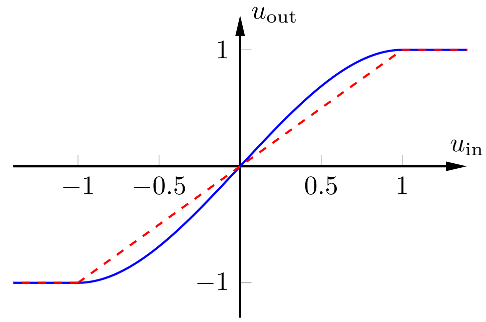

Figure 6 Input-output characteristics of the nonlinear amplifier from Problem 5b (red, dashed) and an ideal saturation (blue).

a. The Nyquist plot should not enter the disk $D(\alpha,\beta) = D(0.5, 5)$ and it should not encircle it. This corresponds to the circle with a real value that is at most $-1/5 = -0.2$. This means that the circle lies outside the Nyquist curve and the feedback connection is BIBO stable.

b. The Nyquist curve of the system $K_pP(s)$ should stay in the interior of a disk $D(\alpha,\beta) = D(-0.5, 0.5)$. This is a circle around the origin with radius 2. Therefore, a proportional gain $K_p < 2$ results in a BIBO stable system.

c. The Nyquist curve should stay in the half-plane $\mathrm{Re}(z) > -1/\beta = -0.1$. We know that

$$K_pP(i\omega) = \frac{K_p}{(i\omega+1)^2} = \frac{K_p(1-\omega^2+2i\omega)}{(\omega^2+1)^2}$$

We are interested in finding the minimum value of the real part of $P(i\omega)$ since it will determine for which β we can guarantee BIBO stability. The real part can be plotted on a graphical calculator to determine that the minimum value of $P(i\omega)$ is $-1/8$. The minimum for $K_pP(i\omega)$ is then $-K_p/8$. Therefore, K_p has to be less than 0.8 for us to conclude BIBO stability with the circle criterion.

5. In this problem we study two saturated nonlinearities, an ideal saturation

$$u_{\text{out}} = \begin{cases} 1 & \text{if} \quad u_{\text{in}} \geq 1 \\ u_{\text{in}} & \text{if} \quad \leq u_{\text{in}} \leq 1 \\ -1 & \text{if} \quad u_{\text{in}} \leq -1 \end{cases} \tag{5}$$

and a saturated amplifier

$$u_{\text{out}} = \begin{cases} 1 & \text{if} \quad u_{\text{in}} \geq 1 \\ (3u_{\text{in}} - u_{\text{in}}^3)/2 & \text{if} \quad -1 \leq u_{\text{in}} \leq 1\,. \\ -1 & \text{if} \quad u_{\text{in}} \leq -1 \end{cases} \tag{6}$$

a. The saturations are shown in Figure 6 and the corresponding describing functions are shown in Figure 7. Identify which describing function corresponds to each nonlinearity and explain the differences and similarities of the describing functions based on the models of the nonlinearities. (1 p)

b. Compute the describing function for the saturating amplifier in Equation (6). *Hint*: $\sin^2(\theta) = (1-\cos(2\theta))/2$ and $\sin^4\theta = (3 - 4\cos 2\theta + \cos 4\theta)/8$, (1.5 p)

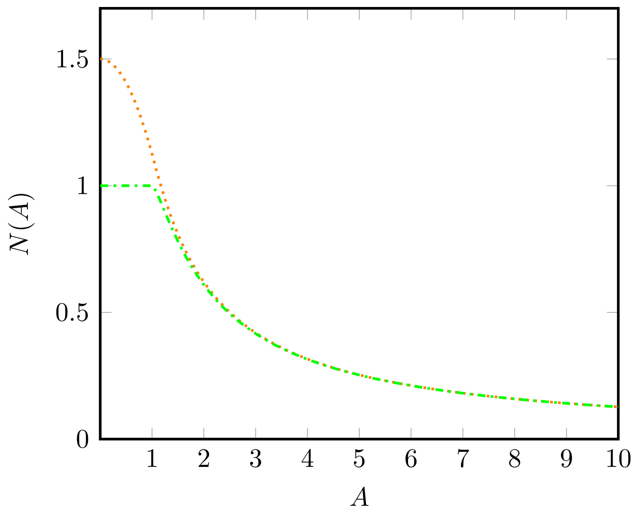

Figure 7 Describing function for saturated amplifier an ideal saturation

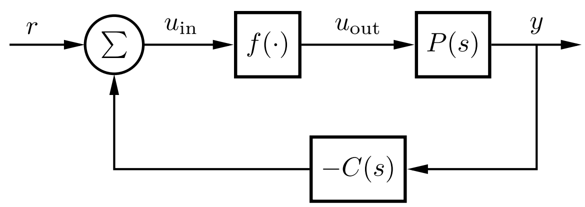

Figure 8 Interconnection of nonlinearity

c. A system $P(s) = 1/(s(s+3))$ is controlled with a controller $C(s) = 38.5/(s+3)$ according to Figure 8. What does the describing method theory predict for each of the two nonlinearities?

Note: It is possible to solve part **c.** without having solved part **b.**.

(2 p)

Solution

a. For low amplitudes (A<1) the output of ideal saturation is $u_{\text{out}} = u_{\text{in}}$, and the describing function is therefore constant 1 here. The saturated amplifier has a higher gain for low amplitudes and for $A \approx 0$, we have $u_{\text{out}} \approx 1.5u_{\text{in}}$. Thus the orange dotted line corresponds to the saturating amplifier, and the green dash-dotted line corresponds to the ideal saturation.

b. We have for $\phi \in [0, \pi/2], u_{\text{out}} = f(u_{\text{in}})$

$$f(A \sin \phi) = \begin{cases} (3A \sin \phi - A^3 \sin^3 \phi)/2 & 0 \le \phi \le \phi_0 \\ 1 & \phi_0 \le \phi \le \pi/2 \end{cases}, \text{ where } \phi_0 = \sin^{-1}(1/A)$$

Since f is odd we have that the imaginary part of the describing function is 0, so for $A < 1$:

$$
\begin{aligned}
N(A) &= \frac{1}{\pi A}\int_0^{2\pi} f(A\sin\phi)\sin\phi \mathrm{d}\phi = \\
&= \frac{2}{\pi A}\int_0^{\phi_0}\Big((3A\sin\phi - A^3\sin^3\phi)\Big)\sin\phi\mathrm{d}\phi + \frac{4}{\pi A}\int_{\phi_0}^{\pi/2}\sin\phi\mathrm{d}\phi
\end{aligned}
$$

Using that $\sin^4(\phi) = (3-4\cos(2\phi)+\cos(4\phi))/8$ and $\sin^2(\phi) = (1-\cos(2\phi))/2$ we get the primitive to the first part

$$
\begin{aligned}
&\int \Big((3A\sin^2\phi - A^3\sin^4\phi)/2\Big)\sin\phi\mathrm{d}\phi = \\
=&\int\left((\frac{3A}{2}(1-\cos(2\phi)) - \frac{A^3}{8}(3-4\cos(2\phi)+\cos(4\phi))) \right)\mathrm{d}\phi = \\
=&\frac{3A}{2}\left(\phi - \frac{1}{2}\sin(2\phi)\right) - \frac{A^3}{8}\left(3\phi - 2\sin(2\phi) + \frac{1}{4}\sin(4\phi)\right)
\end{aligned}
$$

the answer is thus for $A > 1$:

$$
\begin{aligned}
N(A) =&\frac{2}{\pi A}\left[\frac{3A}{2}\left(\phi - \frac{1}{2}\sin(2\phi)\right) - \frac{A^3}{8}\left(3\phi - 2\sin(2\phi) + \frac{1}{4}\sin(4\phi)\right)\right]_0^{\phi_0} + \frac{4}{\pi A}\left[-\cos(\phi)\right]_{\phi_0}^{\pi/2} = \\
=& \frac{3}{\pi}\left(\phi_0 - \frac{1}{2}\sin(2\phi_0)\right) - \frac{A^2}{4\pi}\left(3\phi_0 - 2\sin(2\phi_0) + \frac{1}{4}\sin(4\phi_0)\right) + \frac{4}{A\pi}\cos(\phi_0)
\end{aligned}
$$

and for $|A| \le 1$

$$
\begin{aligned}
N(A) =&\frac{3}{\pi}\left(\pi/2 - \frac{1}{2}\sin(2\pi/2)\right) - \frac{A^2}{4\pi}\left(3\pi/2 - 2\sin(2\pi/2) + \frac{1}{4}\sin(4\pi/2)\right) = \\
=&\frac{3}{8}(4 - A^2)
\end{aligned}
$$

c. Reordering the equations give that $G(s)C(s) = \frac{38.5}{s(s+3)^2}$ can be seen as connected in negative feedback with $f(\cdot)$. The describing function lies on the negative real axis, so we find the intersection with that in the Nyquist diagram.

$$
\begin{aligned}
\arg(G(i\omega)C(i\omega)) = -\pi/2 - 2\mathrm{atan}(w/3) &= -\pi \Leftrightarrow \\
\mathrm{atan}(\omega/3) &= \pi/4 \Leftrightarrow \\
w &= 3
\end{aligned}
$$

we get $|G(s)C(s)| = |\frac{38.5}{w(w^2+p^2)}| = 3/\sqrt{18} \approx 0.713$. I.e. intersection with $-\frac{1}{N(A)}$ occurs when $N(A) \approx 1/0.7 \approx 1.40$. For the ideal saturation, $N(A) \le 1$ so no intersection occurs. $-\frac{1}{N(A)}$ is on the outside of the Nyquist curve in this case so stability is predicted. For the saturaded amplifier, $N(A) = 1.40$ at $A \approx 0.5$ (0.51 to be more exact). The curve $-\frac{1}{N(A)}$ is going from $-2/3$ towards $-\infty$ so in this case a stable oscilation of amplitude 0.5 with frequency 3 is predicted.

6. Consider the following system

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_2 + x_1^2 + u
\end{aligned}
$$

a. Verify that the origin of the system is not globally asymptotically stable for $u = 0$. (1 p)

b. Your goal is to design a sliding mode controller that makes the origin globally asymptotically stable. Based on the functions

$$\sigma_1(x) = x_1 + x_2$$
$$\sigma_2(x) = x_1 - x_2,$$

which of the sliding sets, $\sigma_1(x) = 0$, or $\sigma_2(x) = 0$, guarantee(s) asymptotic convergence of the states to the origin. Motivate your answer by analysis of the dynamics on the set $\{x|\sigma(x) = 0\}$.

(1 p)

c. Design a control law that drives the states of the system to the sliding set you chose above. (2 p)

Solution

a. Linearization around the origin yields:

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} x$$

The linearized system has an eigenvalue in 1, which shows that the origin is unstable.

b. A sliding mode is an invariant set, i.e. $\dot{\sigma} = 0$. We have

$$0 = \dot{\sigma}_1 = \dot{x}_1 + \dot{x}_2 = x_2 + \dot{x}_2 \Rightarrow \dot{x}_2 = -x_2$$

which shows that $x_2(t) \to 0,\ t \to \infty$ along $\sigma_1(x) = 0$. Since $x_1(t) = -x_2(t)$ on the sliding set, also $x_1(t) \to 0,\ t \to \infty$.
For $\sigma_2(x)$ we have:

$$0 = \dot{\sigma}_2 = \dot{x}_1 - \dot{x}_2 = x_2 - \dot{x}_2 \Rightarrow \dot{x}_2 = x_2$$

which means $x_2(t) \to \pm\infty$, along $\sigma_2(x) = 0$.
The answer is: $\sigma_1(x) = 0$ guarantees asymptotic convergence to the origin.

c. Let

$$V(\sigma) = \frac{\sigma^2}{2}$$

We have that:

$$\dot{V} = \frac{dV}{d\sigma}\frac{d\sigma}{dx}\frac{dx}{dt} = \sigma(\dot{x}_1 + \dot{x}_2) = \sigma(2x_2 + x_1^2 + u)$$

By choosing

$$u = -2x_2 - x_1^2 - k\sigma = -(2 + k)x_2 - x_1(x_1 + k)$$

7. Solve the optimal control problem

$$\min \int_0^1 u^2(t)\,dt + x^2(1)$$
$$\frac{d}{dt}x(t) = u \cdot t$$
$$x(0) = 1$$
$$u \in [-\alpha, \alpha]$$

Solve both for the case when the constant α is very large so that the control u does not hit the constraint and for the case when α is so small that it will change the solution. (3 p)

with $k > 0$ we have

$$\dot{V} = -k\sigma^2$$

which means that $\sigma \to 0$ as $t \to \infty$ Another choice is

$$u = -2x_2 - x_1^2 - \mathrm{sign}(\sigma)$$

Solution

Solution:

10. The final time $T = 1$ is fixed. The system is normal so we can put $n_0 = 1$. The Hamiltonian is

$$H = u^2 + \lambda t u$$

Minimization with respect to u gives $u = -sat_\alpha(\lambda(t)t/2)$, where

$$sat_\alpha(x) = \begin{cases} \alpha, & x \geq \alpha \\ x\ , & -\alpha \leq x \leq \alpha \\ -\alpha, & x \leq \alpha \end{cases}$$

The adjoint equation is

$$\dot{\lambda} = -H_x = 0, \quad \lambda(1) = 2x(1) \rightarrow \lambda(t) = 2\,x(1)$$

This gives $u = x(1) \cdot t$. If this is put into the system equation we get

$$x(T) - x(0) = \int_0^T -t^2 x(T)\,dt = x(1) \cdot (-1)^3/3$$

and hence $x(1) = x(0)/(1 + 1^3/3) = \frac{3}{4}$. The optimal control signal is hence

$$u = -\frac{3t}{4}$$

if we do not hit any constraints, that is, if $\alpha > 3 \cdot 1/4$.