LUNDS UNIVERSITET

Lecture 2

FRTN10 Multivariable Control

Automatic Control LTH, 2019

Course Outline

L1–L5 Specifications, models and loop-shaping by hand

1. Introduction
2. **Stability and robustness**
3. Specifications and disturbance models
4. Control synthesis in frequency domain
5. Case study: DVD player

L6–L8 Limitations on achievable performance

L9–L11 Controller optimization: analytic approach

L12–L14 Controller optimization: numerical approach

L15 Course review

L2: Stability and robustness

1. Stability
2. Sensitivity and robustness
3. The Small Gain Theorem
4. Singular values

L2: Stability and robustness

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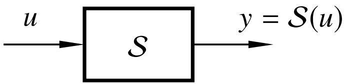
Stability is crucial

Examples:

- bicycle
- JAS 39 Gripen
- Mercedes A-class
- ABS brakes

Input–output stability



A general system $\mathcal{S}$ is called **input–output stable** (or "L_2 stable" or "BIBO stable" or just "stable") if its L_2 gain is finite:

$$\|\mathcal{S}\| = \sup_u \frac{\|\mathcal{S}(u)\|}{\|u\|} < \infty$$

Input–output stability of LTI systems

For an LTI system $\mathcal{S}$ with impulse response $g(t)$ and transfer function $G(s)$, the following stability conditions are equivalent:

- $\|\mathcal{S}\|$ is bounded
- $g(t)$ decays exponentially
- All poles of $G(s)$ are in the left half-plane (LHP), i.e., all poles have negative real part

Internal stability

The LTI system

$$\frac{dx}{dt} = Ax + Bu$$
$$y = Cx + Du$$

is called **internally stable** if the following equivalent conditions hold:

- The state x decays exponentially when $u = 0$
- All eigenvalues of A are in the LHP

Internal vs input–output stability

If

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

is internally stable **then**

$$G(s) = C(sI - A)^{-1}B + D$$

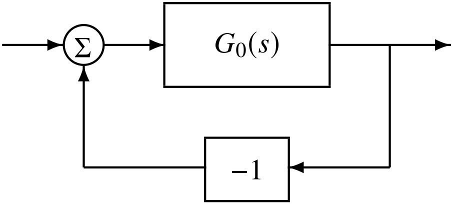
is input–output stable.

Warning

The opposite is not always true! There may be unstable pole-zero cancellations (that also render the system uncontrollable and/or unobservable), and these may not be seen in the transfer function!

Stability of feedback loops

Assume scalar open-loop system $G_0(s)$



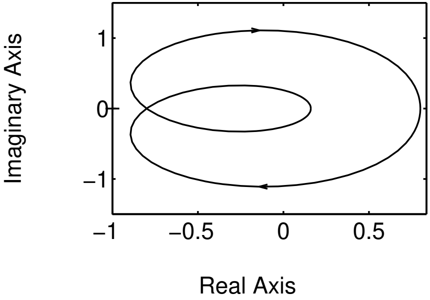
The closed-loop system is stable **if and only if** all solutions to the characteristic equation

$$1 + G_0(s) = 0$$

are in the left half-plane.

Simplified Nyquist criterion

If $G_0(s)$ is stable, then the closed-loop system $[1+G_0(s)]^{-1}$ is stable **if and only if** the Nyquist curve of $G_0(s)$ does not encircle -1.



(Note: Matlab gives a Nyquist plot for both positive and negative frequencies)

General Nyquist criterion

Let

- P = number of **unstable** (RHP) poles in $G_0(s)$
- N = number of **clockwise** encirclements of -1 by the Nyquist plot of $G_0(s)$

Then the closed-loop system $[1 + G_0(s)]^{-1}$ has $P + N$ unstable poles

L2: Stability and robustness

Sensitivity and robustness

- How sensitive is the closed-loop system to model errors and disturbances?
- How do we measure the “distance to instability”?
- Is it possible to guarantee stability for all systems within some distance from the ideal model?

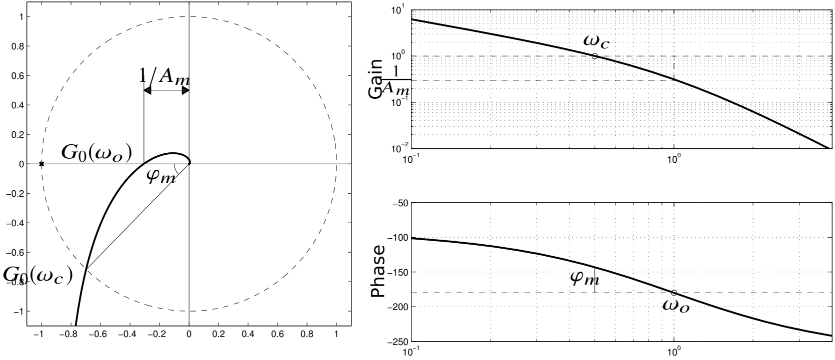
Amplitude and phase margins

Amplitude margin A_m:

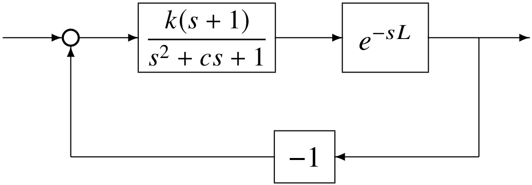
$$\arg G_0(i\omega_0) = -180^\circ, \qquad |G_0(i\omega_0)| = 1/A_m$$

Phase margin φ_m:

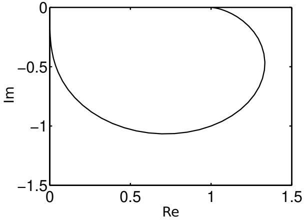
$$|G_0(i\omega_c)| = 1, \qquad \arg G_0(i\omega_c) = \varphi_m - 180^\circ$$

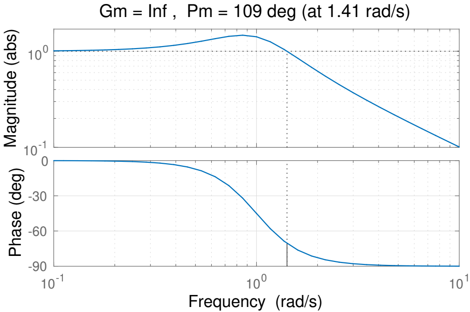


Mini-problem



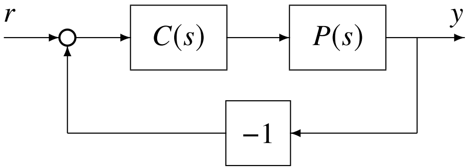
Nominally $k = 1$, $c = 1$ and $L = 0$. How much margin is there in each parameter before the closed-loop system becomes unstable?





Mini-problem

Sensitivity functions



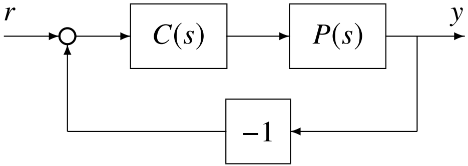
$$S(s) = \frac{1}{1 + P(s)C(s)}$$ sensitivity function

$$T(s) = \frac{P(s)C(s)}{1 + P(s)C(s)}$$ complementary sensitivity function

Note that we always have

$$S(s) + T(s) = 1$$

Sensitivity towards changes in plant



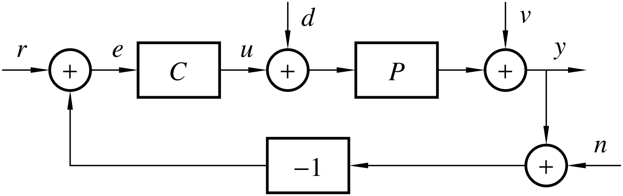
How sensitive is the closed loop to a (small) change in P?

$$\frac{dT}{dP} = \frac{C}{(1+PC)^2} = \frac{T}{P(1+PC)}$$

Relative change in T compared to relative change in P:

$$\frac{dT/T}{dP/P} = \frac{1}{1+PC} = S$$

Sensitivity towards disturbances



Open-loop response ($C = 0$) to process disturbances d, v:

$$Y_{ol} = V + PD$$

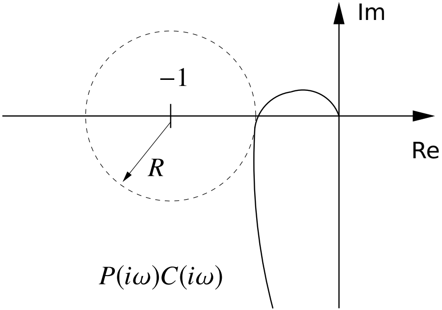
Closed-loop response:

$$Y_{cl} = \frac{1}{1+PC}V + \frac{P}{1+PC}D = S\,Y_{ol}$$

Interpretation as stability margin

The maximum gain of the sensitivity function measures the inverse of the distance between the Nyquist plot and the point -1:

$$R^{-1} = \sup_{\omega} \left| \frac{1}{1 + P(i\omega)C(i\omega)} \right| = M_s$$



L2: Stability and robustness

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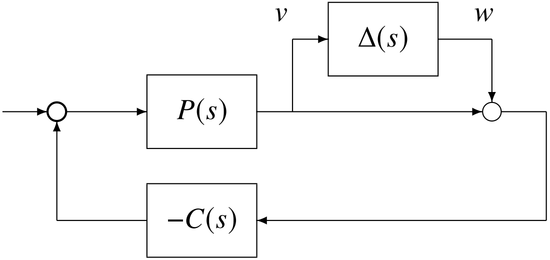
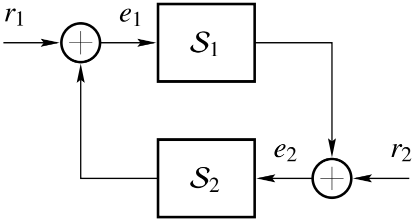
Robustness analysis

How large plant uncertainty Δ can be tolerated without risking instability?

Example (multiplicative uncertainty):



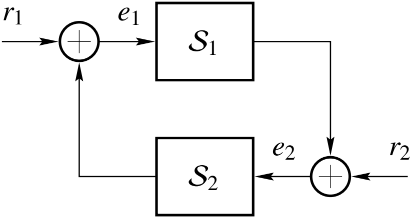
The Small Gain Theorem



Assume that $\mathcal{S}_1$ and $\mathcal{S}_2$ are stable. **If** $\|\mathcal{S}_1\| \cdot \|\mathcal{S}_2\| < 1$, **then** the closed-loop system (from (r_1, r_2) to (e_1, e_2)) is stable.

- Note 1: The theorem applies also to nonlinear, time-varying, and multivariable systems
- Note 2: The stability condition is sufficient but not necessary, so the results may be conservative

The Small Gain Theorem



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Proof

$$e_1 = r_1 + \mathcal{S}_2(r_2 + \mathcal{S}_1(e_1))$$

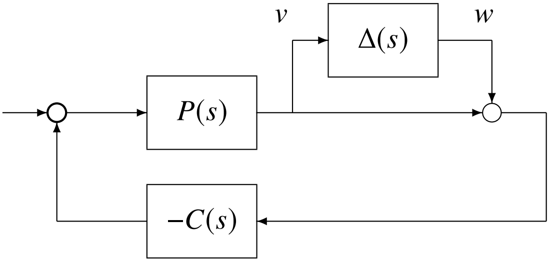
$$\|e_1\| \leq \|r_1\| + \|\mathcal{S}_2\| \Big(\|r_2\| + \|\mathcal{S}_1\| \cdot \|e_1\| \Big)$$

$$\|e_1\| \leq \frac{\|r_1\| + \|\mathcal{S}_2\| \cdot \|r_2\|}{1 - \|\mathcal{S}_1\| \cdot \|\mathcal{S}_2\|}$$

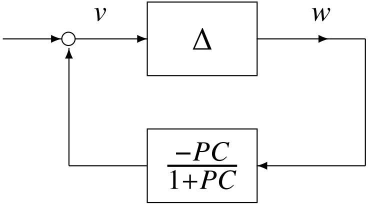
This shows bounded gain from (r_1, r_2) to e_1.

The gain to e_2 is bounded in the same way.

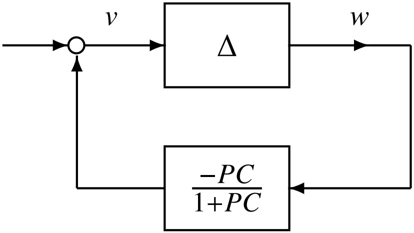
Application to robustness analysis



The diagram can be redrawn as



Application to robustness analysis



Assuming that $T = \frac{PC}{1+PC}$ is stable, the Small Gain Theorem guarantees stability if

$$\|\Delta\| \cdot \|T\| < 1$$

L2: Stability and robustness

Gain of multivariable systems

Recall from Lecture 1 that

$$\|\mathcal{S}\| = \sup_{\omega} |G(i\omega)| = \|G\|_\infty$$

for a stable LTI system $\mathcal{S}$.

How to calculate $|G(i\omega)|$ for a multivariable system?

Vector norm and matrix gain

For a vector $x \in \mathbf{C}^n$, we use the 2-norm

$$|x| = \sqrt{x^* x} = \sqrt{|x_1|^2 + \cdots + |x_n|^2}$$

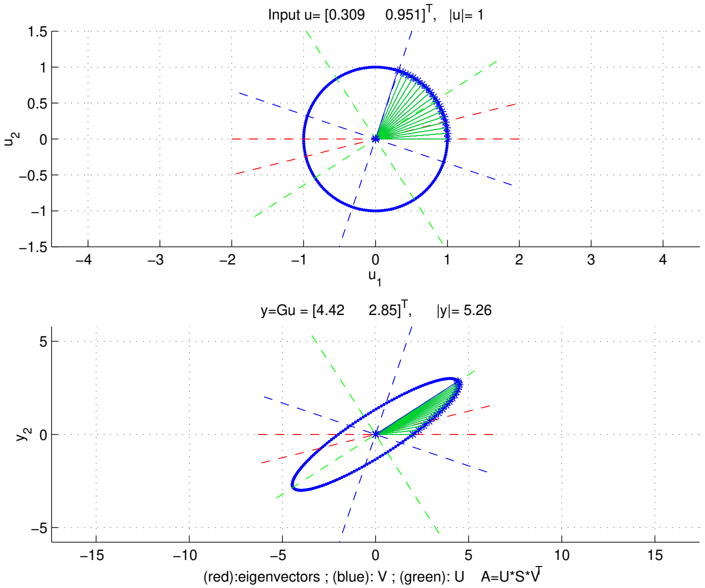
(A^* denotes the conjugate transpose of A)

For a matrix $A \in \mathbf{C}^{n \times m}$, we use the L_2-induced norm

$$\|A\| := \sup_x \frac{|Ax|}{|x|} = \sup_x \sqrt{\frac{x^* A^* A x}{x^* x}} = \sqrt{\overline{\lambda}(A^* A)}$$

$\overline{\lambda}(A^* A)$ denotes the largest eigenvalue of $A^* A$. The ratio $|Ax|/|x|$ is maximized when x is a corresponding eigenvector.

Example: Different gains in different directions: $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$



Singular Values

For a matrix A, its singular values σ_i are defined as

$$\sigma_i = \sqrt{\lambda_i}$$

where λ_i are the eigenvalues of $A^* A$.

Let $\overline{\sigma}(A)$ denote the largest singular value and $\underline{\sigma}(A)$ the smallest singular value.

For a linear map $y = Ax$, it holds that

$$\underline{\sigma}(A) \leq \frac{|y|}{|x|} \leq \overline{\sigma}(A)$$

The singular values are typically computed using singular value decomposition (SVD)

Singular value decomposition (SVD)

Let A be an $m \times n$ complex matrix. It can be factored as

$$A = U\Sigma V^*$$

where

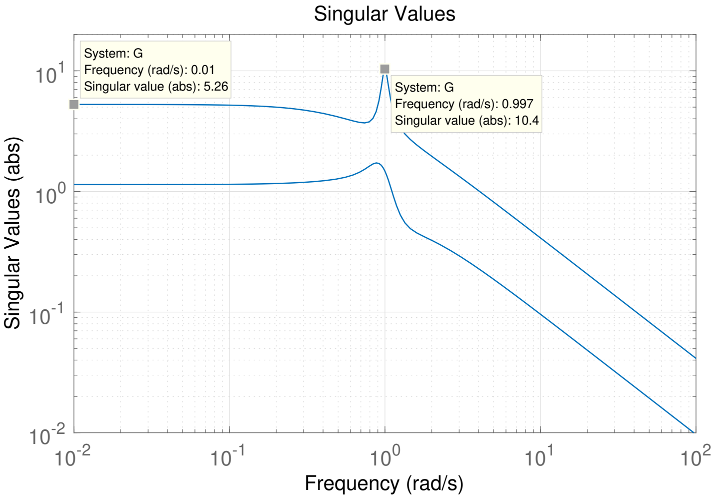
- U is an $m \times m$ unitary complex matrix, whose columns represent different **output directions**
- Σ is an $m \times n$ matrix with non-negative real numbers (the singular values) on the diagonal, representing different **gains**
- V is an $n \times n$ unitary complex matrix, whose columns represent different **input directions**

With $A = G(i\omega)$, the complex directions reveal both the relative magnitude and phase of the input/output signals with frequency ω.

Example: Gain of multivariable system

$$G(s) = \begin{bmatrix} \dfrac{2}{s+1} & \dfrac{4}{2s+1} \\ \dfrac{s}{s^2+0.1s+1} & \dfrac{3}{s+1} \end{bmatrix}$$

```
>> s = zpk('s');
>> G = [2/(s+1) 4/(2*s+1); s/(s^2+0.1*s+1) 3/(s+1)];
>> sigma(G) % plot sigma values of G wrt freq
>> [gain,w] = norm(G,inf)  % infinity norm = system gain
   gain =
      10.3698
   w =
       1.0000
```



The singular values of the transfer function matrix (prev slide). Note that $G(0)$ = [2 4 ; 0 3] (prev example). $\|G\|_\infty = \|G(1i)\| = 10.3698$.

```
>> A = evalfr(G,i*1)
A =
    1.0000 - 1.0000i   0.8000 - 1.6000i
   10.0000 - 0.0000i   1.5000 - 1.5000i

>> [U,S,V] = svd(A)
U =
  -0.1307 + 0.1082i   0.9472 - 0.2720i
  -0.9855 + 0.0023i  -0.1557 - 0.0675i
S =
  10.3698         0
        0    1.4720
V =
  -0.9734 + 0.0000i  -0.2292 + 0.0000i
  -0.1697 - 0.1541i   0.7206 + 0.6544i
```

Summary of Lecture 2

- Input–output stability: $\|\mathcal{S}\| < \infty$
- Sensitivity function: $S(s) := \frac{1}{1+P(s)C(s)}$
  - Three different interpretations
- Small Gain Theorem: The feedback interconnection of $\mathcal{S}_1$ and $\mathcal{S}_2$ is stable **if** $\|\mathcal{S}_1\| \cdot \|\mathcal{S}_2\| < 1$
  - Conservative compared to the Nyquist criterion
  - Useful for robustness analysis
- The gain of a multivariable system $G(s)$ is given by $\sup_\omega \overline{\sigma}(G(i\omega))$, where $\overline{\sigma}$ is the largest singular value
  - Singular values by SVD (on computer): $G(i\omega) = U\Sigma V^*$