LUNDS
UNIVERSITET

Lecture 4

FRTN10 Multivariable Control

Automatic Control LTH, 2019

Course Outline

L1–L5	Specifications, models and loop-shaping by hand
L6–L8	Limitations on achievable performance
L9–L11	Controller optimization: analytic approach
L12–L14	Controller optimization: numerical approach
L15	Course review

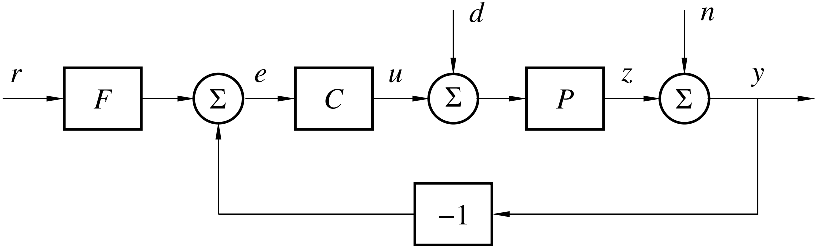
L1–L5 Specifications, models and loop-shaping by hand

1. Introduction
2. Stability and robustness
3. Specifications and disturbance models
4. **Control synthesis in frequency domain**
5. Case study: DVD player

L4: Control synthesis in frequency domain

1. Time and frequency domain specifications
2. Loop shaping
3. Feedforward design

Relations between signals



$$Z = \frac{P}{1+PC}D - \frac{PC}{1+PC}N + \frac{PCF}{1+PC}R$$
$$Y = \frac{P}{1+PC}D + \frac{1}{1+PC}N + \frac{PCF}{1+PC}R$$
$$U = -\frac{PC}{1+PC}D - \frac{C}{1+PC}N + \frac{CF}{1+PC}R$$

Design specifications

Find a controller that

A: reduces the effect of load disturbances

B: does not inject too much measurement noise into the system

C: makes the closed loop insensitive to process variations

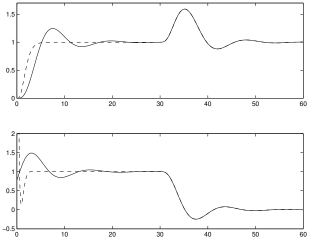
D: makes the output follow the setpoint

Common to have a controller with **two degrees of freedom** (2 DOF), i.e. separate signal transmission from y to u and from r to u. This gives a nice separation of the design problem:

1. Design feedback to deal with A, B, and C
2. Design feedforward to deal with D

Time-domain specifications

- Specifications for deterministic signals, e.g., step response w.r.t. reference change, load disturbance
  - Rise-time T_r
  - Overshoot M
  - Settling time T_s
  - Static error e_0
  - ...



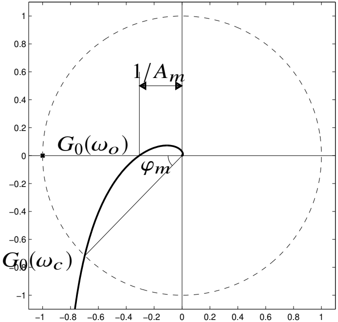
reference step disturbance step

- Stochastic specifications, e.g.,
  - Process output variance
  - Control signal variance

Frequency-domain specifications

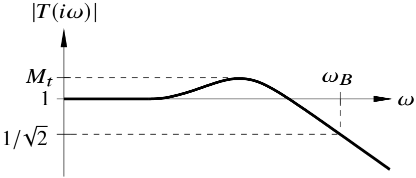
Open-loop specifications (for loop gain $G_0 = L = PC$)

- cross-over frequency ω_c
- phase margin φ_m
- amplitude margin A_m
- . . .



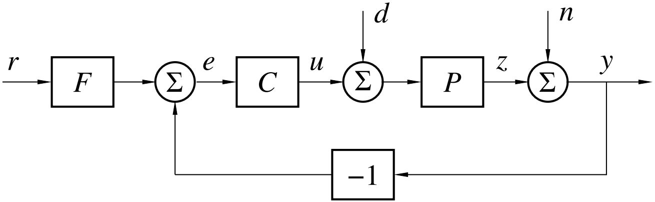
Closed-loop specifications, e.g.

- maximum sensitivity M_s
- resonance peak M_t
- closed-loop bandwidth ω_B
- . . .



Frequency-domain specifications

Closed-loop specifications, cont'd:



Desired properties:

- Small influence of load disturbance d on z $\Leftrightarrow$ $PS \approx 0$
- Limited amplification of noise n in control u $\Leftrightarrow$ $CS \approx 0$
- Small influence of model errors on z $\Leftrightarrow$ $S \approx 0$
- Robust stability despite model errors $\Leftrightarrow$ $T \approx 0$
- Accurate tracking of setpoint r $\Leftrightarrow$ $TF \approx 1$

Frequency domain specifications

$S + T = 1$ and other constraints makes the above impossible to achieve at all frequencies.

Typical design compromise:

- $T \to 0$ for high frequencies ($\omega > \omega_B$)
- $S \to 0$ for low frequencies (+ possibly other disturbance dominated frequencies)

Expressing specifications on S and T

Maximum sensitivity specifications:

- $\|S\|_\infty \leq M_s$
- $\|T\|_\infty \leq M_t$

Typical numbers for M_s and M_t are between 1.0 and 2.0.

Frequency-weighted specifications:

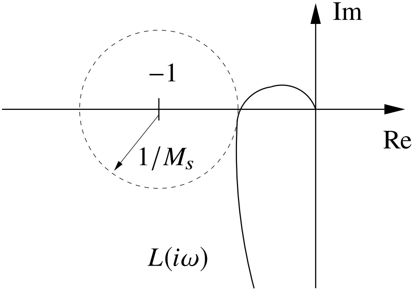
- $\|W_s S\|_\infty \leq 1 \quad \Leftrightarrow \quad |S(i\omega)| \leq |W_s^{-1}(i\omega)|, \ \forall \omega$
- $\|W_t T\|_\infty \leq 1 \quad \Leftrightarrow \quad |T(i\omega)| \leq |W_t^{-1}(i\omega)|, \ \forall \omega$

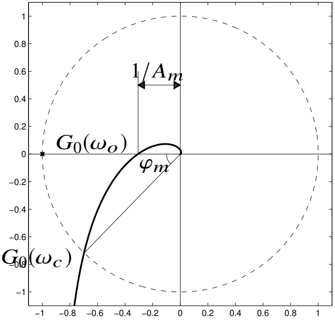
where $W_s(s)$ and $W_t(s)$ are some weighting functions

M_s vs gain and phase margins

Specifying $|S(i\omega)| \leq M_s$ gives bounds for the gain and phase margins (but not the other way around!)

$$|S(i\omega)| \leq M_s \qquad \Rightarrow \qquad A_m \geq \frac{M_s}{M_s - 1}, \quad \varphi_m \geq 2 \arcsin \frac{1}{M_s}$$





Lecture 4 – Outline

1. Time and frequency domain specifications
2. Loop shaping
3. Feedforward design

Loop shaping

Idea: Look at the **loop gain** $L = PC$ for design and translate specifications on S and T into specifications on L

$$S = \frac{1}{1+L} \approx \frac{1}{L} \qquad \text{if } L \text{ is large}$$

$$T = \frac{L}{1+L} \approx L \qquad \text{if } L \text{ is small}$$

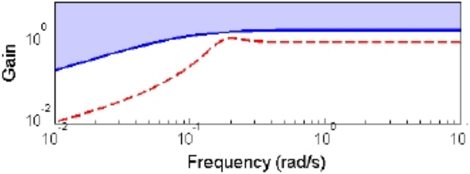
Classical loop shaping: Design C so that $L = PC$ satisfies specifications on S and T

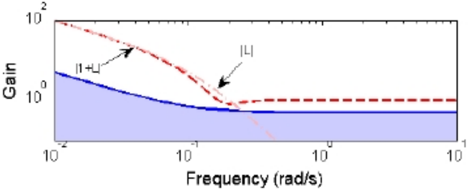
- how are the specifications related?
- what to do with the region around cross-over frequency ω_c (where $|L| \approx 1$)?

Sensitivity vs loop gain

$$S = \frac{1}{1+L}$$

$$|S(i\omega)| \le |W_s^{-1}(i\omega)| \quad \Leftrightarrow \quad |1 + L(i\omega)| > |W_s(i\omega)|$$





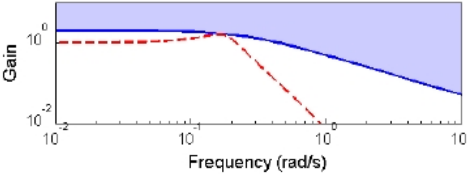
For small frequencies, W_s large and $|L| \approx |1 + L|$.

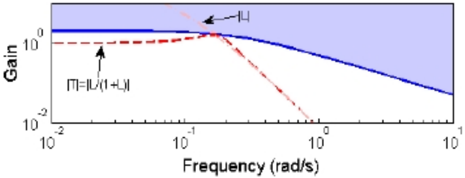
$$|L(i\omega)| \ge |W_s(i\omega)| \quad \text{(approx.)}$$

Complementary sensitivity vs loop gain

$$T = \frac{L}{1+L}$$

$$|T(i\omega)| \leq |W_t^{-1}(i\omega)| \quad \Leftrightarrow \quad \frac{|L(i\omega)|}{|1+L(i\omega)|} \leq |W_t^{-1}(i\omega)|$$

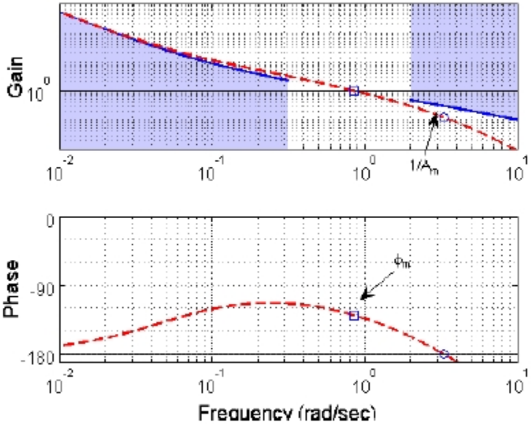




For large frequencies, W_t^{-1} small and $\frac{|L|}{|1+L|} \approx |L|$

$$|L(i\omega)| \leq |W_t^{-1}(i\omega)| \quad \text{(approx.)}$$

Resulting constraints on loop gain L:



Approximations are inexact around cross-over frequency ω_c. In this region, focus is on stability margins (A_m, φ_m)

From requirements to loop shaping – summary

Map specifications on requirements on loop gain L.

- Low-frequency specifications from W_s: $|L| \geq |W_s|$
- High-frequency specifications from W_t: $|L| \leq |W_t^{-1}|$
- Around cross-over frequency, mapping is crude
  - Position cross-over frequency (constrained by W_s, W_t)
  - Adjust phase margin (e.g. from M_s, M_t specifications)

Lead–lag compensation

Shape the loop gain $L = PC$ using a compensator $C = C_1 C_2 C_3 \cdots$ composed of various elements (links), such as

- gain

$$K$$

- lag (phase retarding) elements

$$C_{lag}(s) = \frac{s+a}{s+a/M}, \quad M > 1$$

- lead (phase advancing) elements

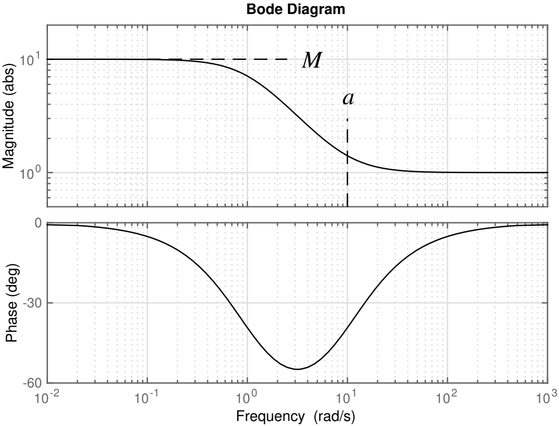
$$C_{lead}(s) = N \frac{s+b}{s+bN}, \quad N > 1$$

Example:

$$C(s) = K \frac{s+a}{s+a/M} \cdot N \frac{s+b}{s+bN}$$

Lag filter

$$C_{lag}(s) = \frac{s+a}{s+a/M}, \quad M > 1$$

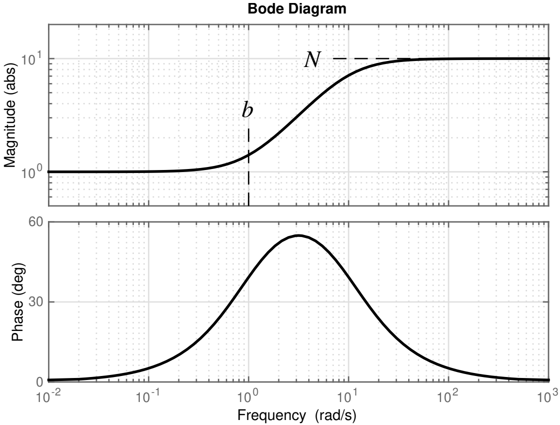


Lag filter

- Increases low-frequency loop gain by factor M
  - $M = \infty \implies$ PI controller
  - Reduces static error by factor M if $L(s)$ contains an integrator
- Break frequency a should be as high as possible for fast disturbance rejection, but too high a reduces stability margins
  - Rule of thumb $a = 0.1\omega_c$ guarantees that φ_m is reduced less than 6°

Lead filter

$$G_{lead}(s) = N\frac{s+b}{s+bN}, \quad N > 1$$



Lead filter

- Increases phase by amount that depends on N (see Collection of Formulae), maximum phase lead at $\omega = b\sqrt{N}$
  - Typically placed at desired cross-over frequency ω_c
- Gain at $\omega = b\sqrt{N}$ increases by $\sqrt{N}$. To retain the same cross-over frequency, the overall controller gain must be decreased

Iterative lead–lag design

Typical workflow:

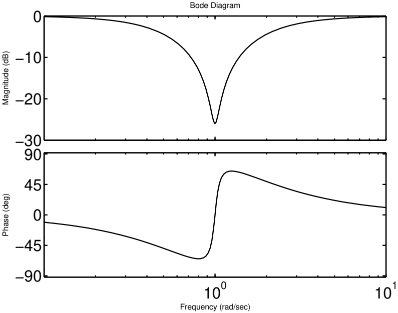
- Adjust gain to obtain the desired cross-over frequency
- Add lag element to improve the low-frequency gain
- Add lead element to improve the phase margin

Adding a lead element while retaining the cross-over frequency affects the low-frequency gain

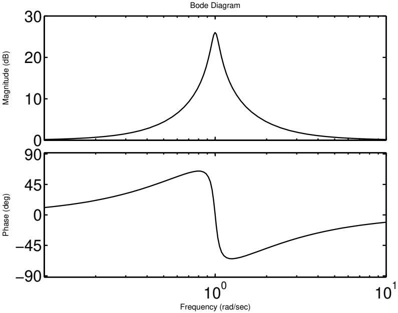
Need to iterate! (Example in Lecture 5)

Links with complex poles/zeros

Example (notch/resonance filters): $\dfrac{s^2 + 2\zeta_a\omega_0 s + \omega_0^2}{s^2 + 2\zeta_b\omega_0 s + \omega_0^2}$



$\omega_0 = 1,\ \zeta_a = 0.05,\ \zeta_b = 1$



$\omega_0 = 1,\ \zeta_a = 1,\ \zeta_b = 0.05$

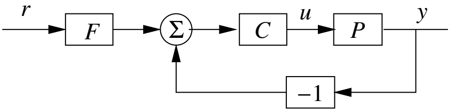
Lecture 4 – Outline

1. Time and frequency domain specifications
2. Loop shaping
3. Feedforward design

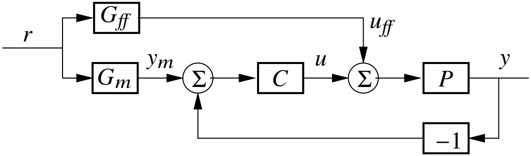
Feedforward design

Two common (and equivalent) 2-DOF configurations:

(1)

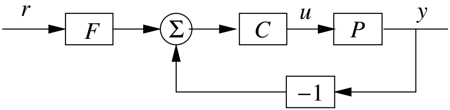


(2)



Ideally, we would like the output to follow the setpoint perfectly, i.e., $y(t) = r(t)$

Feedforward design (1)



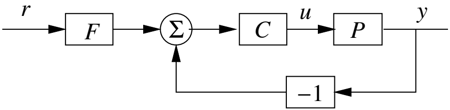
Perfect following would require

$$F = \frac{1+PC}{PC} = T^{-1}$$

This is in general impossible because of pole excess in T (which leads to infinite high-frequency gain in F). Also

- T might contain non-minimum-phase factors (RHP zeros and time delays) that must not be inverted
- u must typically satisfy some upper and lower limits

Feedforward design (1)

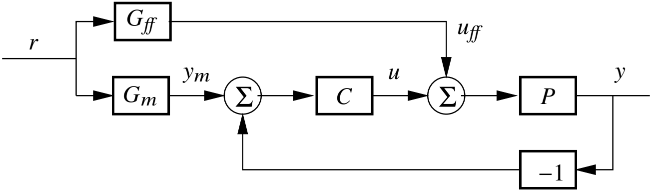


Assume T minimum phase. An implementable choice of F is then

$$F = \frac{T^{-1}}{(sT_f + 1)^d} = \frac{1 + PC}{PC(sT_f + 1)^d}$$

where d is large enough to make F proper

Feedforward design (2)

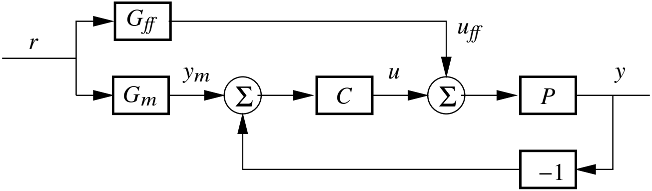


G_m and G_{ff} can be viewed as generators of the desired output y_m and the feedforward u_{ff} that corresponds to y_m

For y to follow y_m, select

$$G_{ff} = \frac{G_m}{P}$$

Feedforward design (2)



Since $G_{ff} = G_m/P$ should be stable, causal and proper we find that

- RHP zeros and time delays of P must be included in G_m
- The pole excess of G_m must not be smaller than the pole excess of P

Take process limitations into account!

Feedforward design – example

Process:

$$P(s) = \frac{1}{(s+1)^4}$$

Selected reference model:

$$G_m(s) = \frac{1}{(sT_m+1)^4}$$

Then

$$G_{ff}(s) = \frac{G_m(s)}{P(s)} = \frac{(s+1)^4}{(sT_m+1)^4} \qquad G_\infty(\infty) = \frac{1}{T_m^4}$$

Fast response (small T_m) requires high high-frequency gain in G_{ff}.

Bounds on the control signal limit how fast response we can obtain in practice

Summary of Lecture 4

Frequency domain design / loop shaping:

- Good mapping between S, T and $L = PC$ at low and high frequencies; mapping around ω_c less clear
- Simple relation between C and $L \;\Rightarrow\;$ "easy" to shape L
- Lead–lag design: iterative procedure to design $C = C_1C_2C_3\cdots$ by hand (topic of Lab 1)

Feedforward design

- Involves taking the inverse of the plant model
- Must respect RHP zeros, time delays and pole excess
  - Same rules apply to Internal Model Control design, see Lecture 12