LUNDS UNIVERSITET

Lecture 12

FRTN10 Multivariable Control

Automatic Control LTH, 2019

Course Outline

L1–L5	Specifications, models and loop-shaping by hand
L6–L8	Limitations on achievable performance
L9–L11	Controller optimization: analytic approach
L12–L14	Controller optimization: numerical approach

12. **Youla parametrization, internal model control**
13. Synthesis by convex optimization
14. Controller simplification, course review

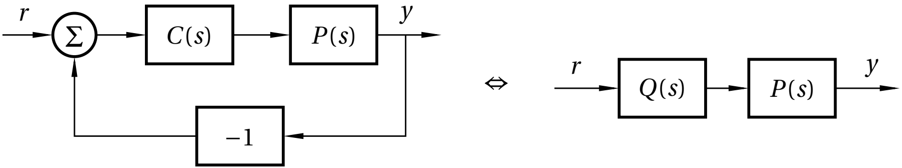
L15	Course review

L12: Youla parametrization, internal model control

1. The Youla (Q) parameterization
2. Internal model control (IMC)

Basic idea of Youla and IMC

Assume stable SISO plant P. Model for design:

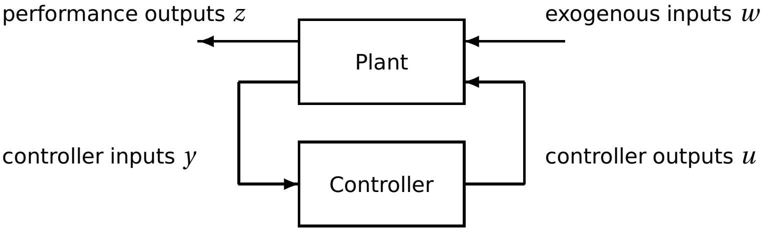


$$\frac{PC}{1+PC} = PQ$$

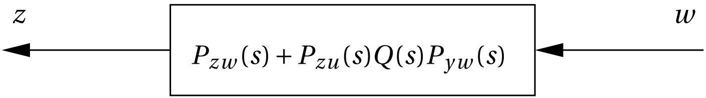
$$Q = \frac{C}{1+PC}$$

Design Q to get desired closed-loop properties. Then $C = \dfrac{Q}{1-QP}$

General idea for Lectures 12–14



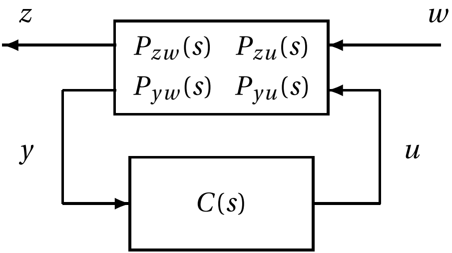
The choice of controller corresponds to designing a transfer matrix $Q(s)$, to get desirable properties of the following map from w to z:



Once $Q(s)$ has been designed, the corresponding controller can be found.

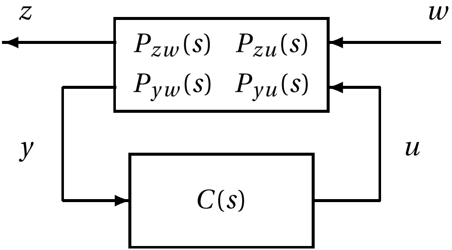
The Youla (Q) parameterization

General feedback control system (assuming positive feedback!):



$$
\begin{aligned}
Z(s) &= P_{zw}(s)W(s) + P_{zu}(s)U(s) \\
Y(s) &= P_{yw}(s)W(s) + P_{yu}(s)U(s) \\
U(s) &= C(s)Y(s)
\end{aligned}
$$

The Youla (Q) parameterization



Closed-loop transfer function from w to z:

$$G_{zw}(s) = P_{zw}(s) + P_{zu}(s)\underbrace{C(s)\left[I - P_{yu}(s)C(s)\right]^{-1}}_{=Q(s)}P_{yw}(s)$$

Given $Q(s)$, the controller is $C(s) = \left[I + Q(s)P_{yu}(s)\right]^{-1}Q(s)$

All stabilizing controllers

Suppose the plant $P = \begin{bmatrix} P_{zw} & P_{zu} \\ P_{yw} & P_{yu} \end{bmatrix}$ is stable. Then

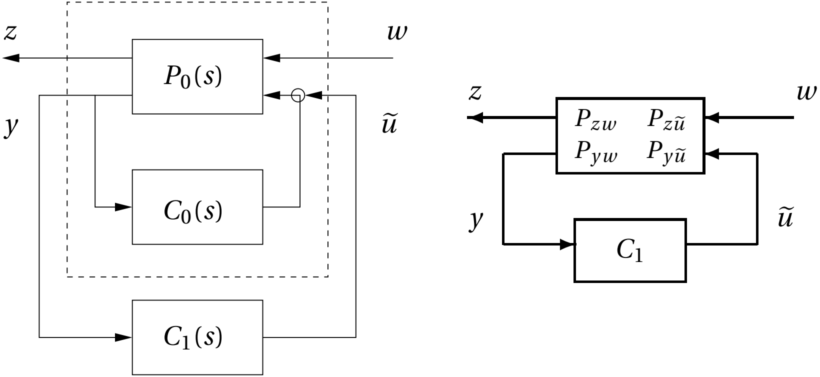
- Stabilty of Q implies stability of $P_{zw} + P_{zu} Q P_{yw}$
- If $Q = C\left[I - P_{yu} C\right]^{-1}$ is unstable, then the closed loop is unstable.

Hence, if P is stable then **all stabilizing controllers** are given by

$$C(s) = \left[I + Q(s) P_{yu}(s)\right]^{-1} Q(s)$$

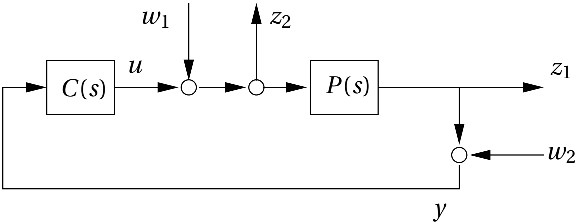
where $Q(s)$ is an arbitrary stable transfer function.

Dealing with unstable plants



If $P_0(s)$ is unstable, let $C_0(s)$ be some stabilizing controller. Then the previous argument can be applied with P_{zw}, $P_{z\tilde{u}}$, P_{yw}, and $P_{y\tilde{u}}$ representing the stabilized system.

Example – DC-motor



Assume we want to optimize the closed-loop transfer matrix from $(w_1, w_2)^T$ to $(z_1, z_2)^T$,

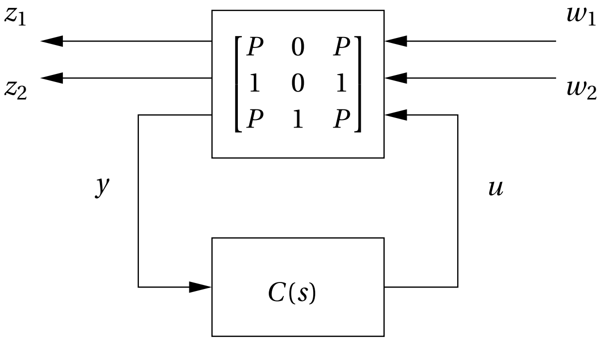
$$G_{zw}(s) = \begin{bmatrix} \frac{P}{1-PC} & \frac{PC}{1-PC} \\ \frac{1}{1-PC} & \frac{C}{1-PC} \end{bmatrix}$$

when $P(s) = \frac{20}{s(s+1)}$.

Find the Youla parameterization of all stable closed-loop systems $G_{wz}(s)$ and the corresponding stabilizing controllers $C(s)$.

Stabilizing controller for DC-motor

Generalized plant model:

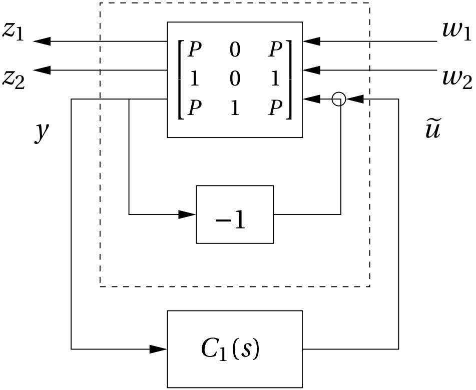


$P(s) = \frac{20}{s(s+1)}$ is not stable, so introduce

$$C(s) = C_0(s) + C_1(s)$$

where $C_0(s) = -1$ stabilizes the plant; $P_c(s) = \frac{P(s)}{1+P(s)} = \frac{20}{s^2+s+20}$

Redrawn diagram for DC-motor example

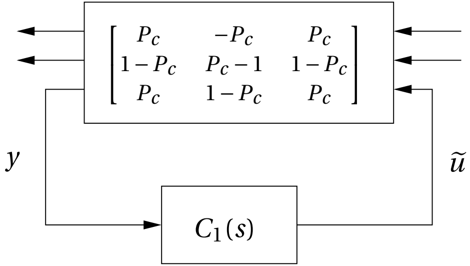


$$z_1 = Pw_1 + P(\tilde{u} - y)$$

$$z_2 = w_1 + \tilde{u} - y$$

$$y = Pw_1 + w_2 + P(\tilde{u} - y) \quad \Rightarrow \quad y = \frac{P}{1+P}w_1 + \frac{1}{1+P}w_2 + \frac{P}{1+P}\tilde{u}$$

Redrawn diagram for DC-motor example



All stable closed-loop systems are parameterized by

$$G_{zw} = \underbrace{\begin{bmatrix} P_c & -P_c \\ 1-P_c & P_c-1 \end{bmatrix}}_{P_{zw}} + \underbrace{\begin{bmatrix} P_c \\ 1-P_c \end{bmatrix}}_{P_{z\tilde{u}}} Q \underbrace{\begin{bmatrix} P_c & 1-P_c \end{bmatrix}}_{P_{yw}}$$

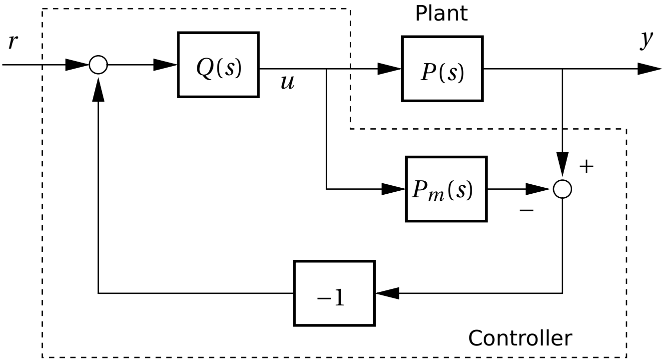
where $Q(s)$ is any stable transfer function.

The controllers are given by $C(s) = C_0(s) + C_1(s) = -1 + \frac{Q(s)}{1+Q(s)P_c(s)}$

L12: Youla parametrization, internal model control

1. The Youla (Q) parameterization
2. Internal model control (IMC)

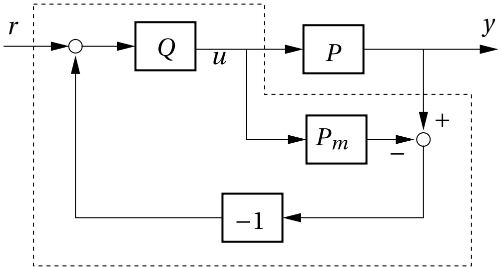
Internal model control (IMC)

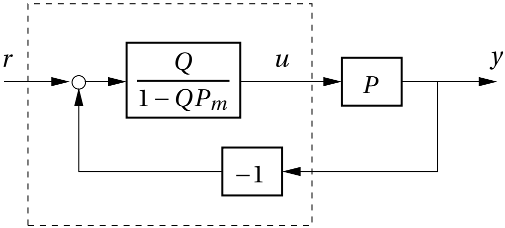


(Negative) Feedback is used only if the real plant $P(s)$ deviates from the model $P_m(s)$. $Q(s)$, $P(s)$, $P_m(s)$ must be stable.

If $P_m(s) = P(s)$, the transfer function from r to y is $P(s)Q(s)$.

Two equivalent diagrams





IMC design rules

With $P(s) = P_m(s)$, the transfer function from r to y is $P(s)Q(s)$.

For perfect reference following, one would like to have $Q(s) = P^{-1}(s)$, but that is is not possible

Design rules:

1. If $P(s)$ is strictly proper, the inverse would have more zeros than poles. Instead, one can choose

$$Q(s) = \frac{1}{(\lambda s + 1)^n} P^{-1}(s)$$

where n is large enough to make Q proper. The parameter λ determines the speed of the closed-loop system.

(cf. feedforward design in Lecture 4)

IMC design rules

2. If $P(s)$ has an unstable zero, the inverse would be unstable. Two different options:
    - Remove the unstable factor $(-\beta s + 1)$ from the plant numerator before inverting.
    - Replace the unstable factor $(-\beta s + 1)$ with $(\beta s + 1)$. With this option, only the phase is modified, not the amplitude function.
3. If $P(s)$ includes a time delay, its inverse would be non-causal. Instead, the time delay is removed before inverting.

IMC design example 1 — first-order plant

$$P(s) = \frac{1}{Ts+1}$$

$$Q(s) = \frac{1}{\lambda s+1} P(s)^{-1} = \frac{Ts+1}{\lambda s+1}$$

$$C(s) = \frac{Q(s)}{1-Q(s)P(s)} = \frac{\frac{Ts+1}{\lambda s+1}}{1-\frac{1}{\lambda s+1}} = \underbrace{\frac{T}{\lambda}\left(1+\frac{1}{sT}\right)}_{\text{PI controller}}$$

Note that $T_i = T$

This way of tuning a PI controller is known as *lambda tuning*

IMC design example 2 — non-minimum phase plant

$$P(s) = \frac{-\beta s + 1}{Ts + 1}, \quad \beta > 0$$

$$Q(s) = \frac{(-\beta s + 1)}{(\beta s + 1)} P(s)^{-1} = \frac{Ts + 1}{\beta s + 1}$$

$$C(s) = \frac{Q(s)}{1 - Q(s)P(s)} = \frac{\frac{Ts+1}{\beta s+1}}{1 - \frac{(-\beta s+1)}{(\beta s+1)}} = \underbrace{\frac{T}{2\beta}\left(1 + \frac{1}{sT}\right)}_{\text{PI controller}}$$

Note that, again, $T_i = T$

The gain is adjusted in accordance with the fundamental limitation imposed by the RHP zero in $1/\beta$.

IMC design for deadtime processes

Consider the deadtime process

$$P = P_0 e^{-sL}$$

where the delay L is assumed known and constant.

Let $C_0 = Q/(1 - QP_0)$ be a controller designed for the delay-free plant model P_0. Solving for Q gives

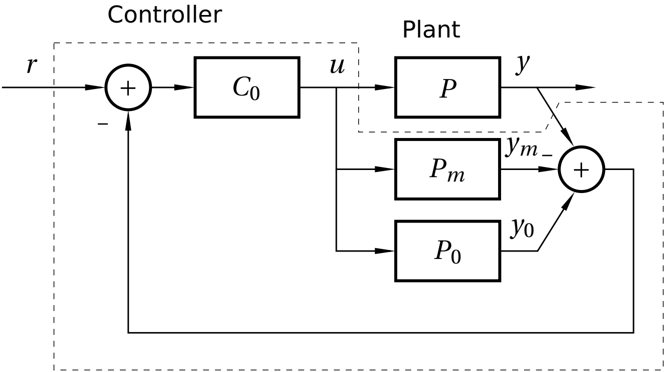
$$Q = \frac{C_0}{1 + C_0 P_0}$$

The controller then becomes

$$C = \frac{Q}{1 - QP_0 e^{-sL}} = \frac{C_0}{1 + (1 - e^{-sL}) C_0 P_0}$$

This modification of C_0 to account for a time delay is known as a Smith predictor.

Smith predictor



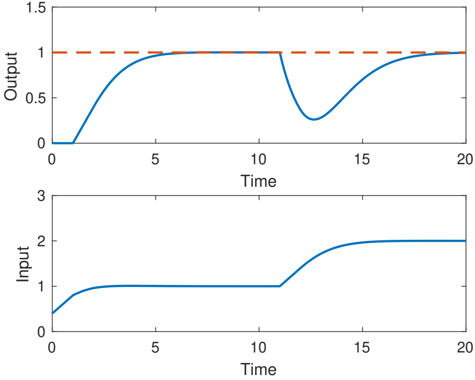
Ideally y and y_m cancel each other and only feedback from y_0 “without delay” is used. If $P = P_m$ then

$$Y = \frac{C_0 P_0}{1 + C_0 P_0} e^{-sL} R$$

Smith predictor – example

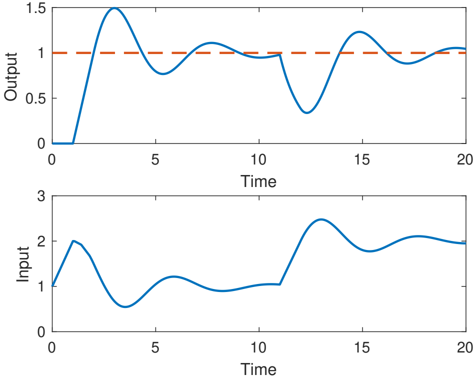
Plant: $P(s) = \dfrac{1}{s+1} e^{-s}$, nominal controller: $C_0(s) = K\left(1 + \dfrac{1}{s}\right)$

Simulation with $K = 0.4$, no Smith predictor ($M_s = 1.4$):



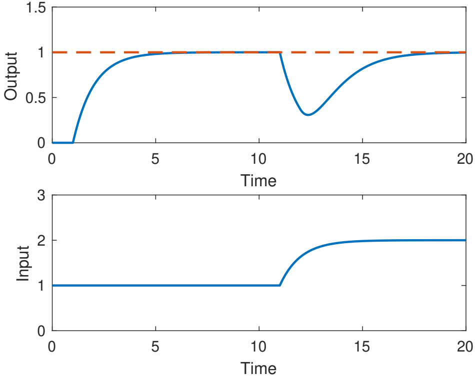
Smith predictor – example

Simulation with $K = 1$, no Smith predictor ($M_s = 3.1$):



Smith predictor – example

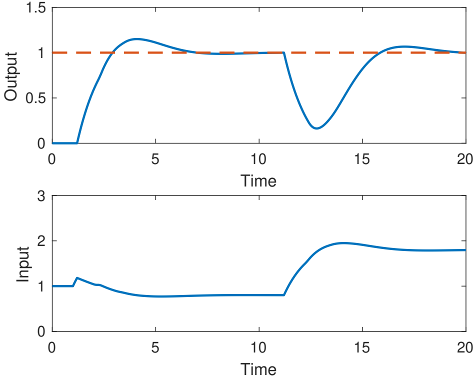
Simulation with $K = 1$ with Smith predictor ($P_m(s) = P(s)$, $M_s = 1.5$):



Looks perfect. But do not the forget the fundamental limitation imposed by the time delay! Respect the rule of thumb $\omega_c < \frac{1.6}{L}$ when designing C_0.

Smith predictor – example

Simulation with $K = 1$ with Smith predictor as before and true process $P(s) = \frac{1}{s+0.8} e^{-1.2s}$



Some performance degradation due to model and plant mismatch.

Lecture 12 – summary

- Idea: Parameterize the closed loop as

$$G_{yr} = PQ$$ SISO case, for IMC design

or

$$G_{zw} = P_{zw} + P_{zu} Q P_{yw}$$ General MIMO case, suitable for optimization

for some stable Q.

- After designing Q, the controller is given by

$$C = \frac{Q}{1 - QP}$$ SISO case (assuming negative feedback)

or

$$C = \left[I + QP_{yu} \right]^{-1} Q$$ General MIMO case (positive feedback)