FRTN10 Exercise 3. Specifications and Disturbance Models

3.1 A feedback system is shown in Figure 3.1, in which a first-order process if controlled by an I controller.

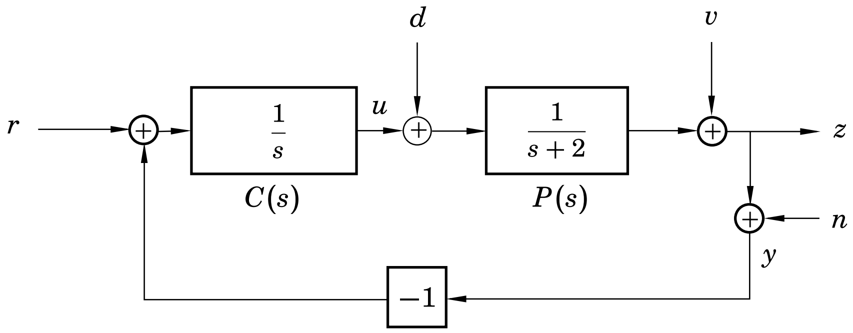


Figure 3.1 System in Problem 3.1.

a. Verify that the closed-loop system is stable.

b. Sketch the Bode amplitude diagrams of the "Gang of Four" for the feedback system:

$$\frac{PC}{1+PC} = T, \quad \frac{P}{1+PC} = PS, \quad \frac{C}{1+PC} = CS, \quad \frac{1}{1+PC} = S$$

Based on the diagrams, answer the following questions:

- Up to approximately what frequency can the process output track the reference value?
- Can the feedback system reject a constant input load disturbance?
- What is the maximum amplification from measurement noise to the control signal?

c. Calculate how much the process output z will vary if the disturbance v is a sinusoidal with amplitude 1 and frequency 0.5 rad/s.

d. Extend the block diagram to explicitly model that v is a sinusoidal disturbance with frequency 0.5 rad/s. (The exact amplitude and phase of the disturbance is not important here.)

3.2 A continuous-time stochastic process $y(t)$ has the power spectrum $\Phi_y(\omega)$. The process can be represented by a linear filter $G(s)$ that has unit-intensity white noise v as input. Determine the linear filter when

a.

$$\Phi_y(\omega) = \frac{a^2}{\omega^2 + a^2}, \quad a > 0$$

b.

$$\Phi_y(\omega) = \frac{a^2 b^2}{(\omega^2 + a^2)(\omega^2 + b^2)}, \quad a, b > 0$$

3.3 A linear system with two inputs and one output has the state-space description

$$\dot{x} = \begin{bmatrix} -5 & -3 \\ 2 & 0 \end{bmatrix} x + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 3 \end{bmatrix} x$$

Assuming that u_1 and u_2 are independent, zero-mean, unit intensity white noise processes, calculate the stationary variance of y.

3.4 Consider a missile travelling in the air. It is propelled forward by a jet force u along a horizontal path. The coordinate along the path is z. We assume that there is no gravitational force. The aerodynamic friction force is described by a simple model as

$$f = k_1 \dot{z} + v,$$

where v are random variations due to wind and pressure changes. Combining this with Newton's second law, $m\ddot{z} = u - f$, where m is the mass of the missile, gives the input-output relation

$$\ddot{z} + \frac{k_1}{m}\dot{z} = \frac{1}{m}(u - v).$$

a. Express the input-output relation in state-space form.

b. The disturbance v has been determined to have the spectral density

$$\Phi_v(\omega) = k_0 \frac{1}{\omega^2 + a^2}, \qquad k_0, a > 0$$

Expand your state-space description so that the disturbance input can be expressed as white noise.

3.5 (*) This problem builds on Problem 3.4.

a. Assume that the position measurement is distorted by an additive error $n(t)$,

$$y(t) = z(t) + n(t)$$

Write down the state-space equations for the system, assuming that $n(t)$ is white noise with intensity 0.1, i.e. $\Phi_n(\omega) = 0.1$.

b. Solve the same problem, this time with

$$\Phi_n(\omega) = 0.1 \frac{\omega^2}{\omega^2 + b^2}, \qquad b > 0$$

c. Solve the problem with

$$\Phi_n(\omega) = 0.1 \frac{1}{\omega^2 + b^2}, \qquad b > 0$$

3.6 (*) Consider an electric motor with the transfer function

$$G(s) = \frac{1}{s(s+1)}$$

from input current to output angle.

There are two different disturbance scenarios:

(i) $\quad Y(s) = G(s)(U(s) + W(s))$

(ii) $\quad Y(s) = G(s)U(s) + W(s)$

In both cases, $\dot{w}(t) = v(t)$, where $v(t)$ is a unit disturbance, e.g., an impulse.

a. Draw block diagrams of the two cases.

b. Convert both cases into state-space form.

c. Give a physical interpretation of $w(t)$ in both cases.

Solutions to Exercise 3. Specifications and Disturbance Models

3.1 **a.** The closed-loop transfer function from r to y is given by

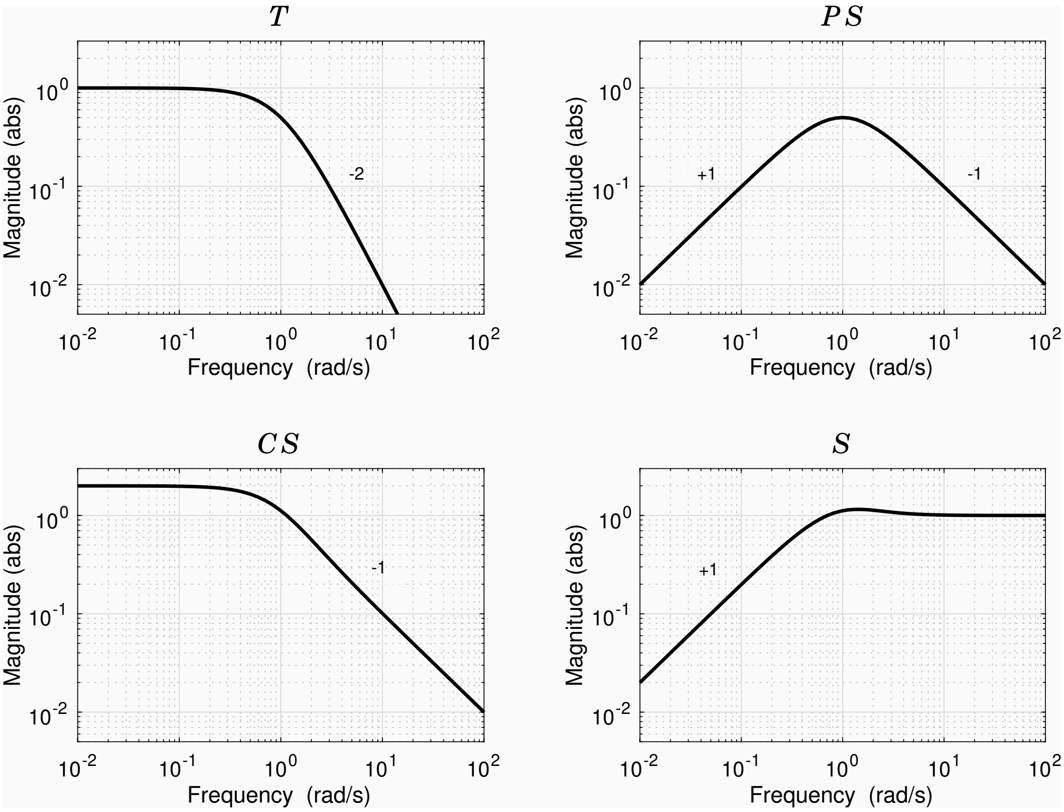
$$T = \frac{1}{(s+1)^2}$$

with two LHP poles in -1.

b. The other three closed-loop transfer functions are

$$PS = \frac{s}{(s+1)^2}, \quad CS = \frac{s+2}{(s+1)^2}, \quad S = \frac{s(s+2)}{(s+1)^2}$$

The four Bode amplitude diagrams are plotted below.



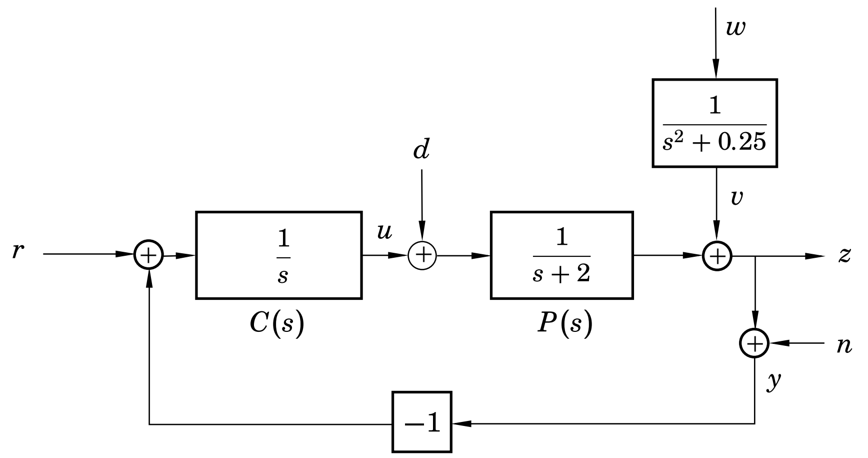
- From the T plot, we see that z can track r up to approx. $\omega = 1$ rad/s.
- Yes, from the PS plot, we see that the gain from d to z approaches 0 when $\omega \to 0$.
- From the CS plot, we see that the maximum gain from n to u is 2.

c. The gain from n to z at $\omega = 0.5$ rad/s is given by

$$|S(i0.5)| = \frac{0.5\sqrt{0.5^2 + 2^2}}{0.5^2 + 1^2} = 0.8246$$

The output z will hence be a sinusoidal with the amplitude 0.8246.

d. The sinusoidal signal can be generated by a system with poles in $\pm 0.5i$, e.g., $\dfrac{1}{s^2+0.25}$, see below.



3.2 $\Phi_y(\omega)$ is an even, scalar, non-negative function. Thus we can factor it into

$$\Phi_y(\omega) = G(i\omega)\Phi_v(\omega)G(-i\omega)$$

where $G(s)$ has its poles and zeroes in the left half-plane and $\Phi_v(\omega) = 1$ (white noise).

a.

$$\Phi_y(\omega) = \frac{a^2}{\omega^2+a^2}\Phi_e(\omega) = \frac{a}{i\omega+a}\cdot\frac{a}{-i\omega+a}$$

So the linear filter is

$$G(s) = \frac{a}{s+a}$$

b. In the same way, we get

$$\Phi_y(\omega) = \frac{a^2b^2}{(\omega^2+a^2)(\omega^2+b^2)}\Phi_e(\omega)$$
$$= \frac{ab}{(i\omega+a)(i\omega+b)}\cdot\frac{ab}{(-i\omega+a)(-i\omega+b)}$$
$$\Rightarrow \quad G(s) = \frac{ab}{(s+a)(s+b)}$$

3.3 Let $\Pi_x = \begin{bmatrix}\pi_{11} & \pi_{12}\\ \pi_{12} & \pi_{22}\end{bmatrix}$ be the stationary state covariance $\mathrm{E}\,xx^T$. Since the system is stable (the A-matrix has eigenvalues $\lambda_1 = -3$, $\lambda_2 = -2$), Π_x is given by the Lyapunov equation

$$A\Pi_x + \Pi_x A^T + BRB^T = 0$$

$$\begin{bmatrix}-5 & -3\\ 2 & 0\end{bmatrix}\begin{bmatrix}\pi_{11} & \pi_{12}\\ \pi_{12} & \pi_{22}\end{bmatrix} + \begin{bmatrix}\pi_{11} & \pi_{12}\\ \pi_{12} & \pi_{22}\end{bmatrix}\begin{bmatrix}-5 & 2\\ -3 & 0\end{bmatrix} + \begin{bmatrix}4 & 0\\ 0 & 4\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

with the solution

$$\Pi_x = \begin{bmatrix} 1 & -1 \\ -1 & \frac{7}{3} \end{bmatrix}$$

The output has the variance

$$\mathrm{E}\, y^2 = \mathrm{E}\,(Cx)(Cx)^T = C\,\mathrm{E}(xx^T)\,C^T = C\Pi_x C^T = 21$$

3.4 a. To make a state-space description, we let, e.g., $x_1 = z$, $x_2 = \dot{z} \quad \Longrightarrow$

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = \frac{1}{m}(u - k_1 x_2 - v).$$

In matrix form:

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & -\frac{k_1}{m} \end{pmatrix} x + \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} u + \begin{pmatrix} 0 \\ -\frac{1}{m} \end{pmatrix} v,$$
$$z = \begin{pmatrix} 1 & 0 \end{pmatrix} x.$$

b. We want to find a filter H such that

$$\Phi_v(\omega) = |H(i\omega)|^2 \Phi_e(\omega)$$

Thus $H(s) = \frac{\sqrt{k_0}}{s+a}$, which is equivalent to $\dot{v} + av = \sqrt{k_0}\, e$.
Adding a new state $x_3 = v$ to the state-space description, gives

$$\dot{x}_3 = -ax_3 + \sqrt{k_0} e$$

and

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -\frac{k_1}{m} & -\frac{1}{m} \\ 0 & 0 & -a \end{pmatrix} x + \begin{pmatrix} 0 \\ \frac{1}{m} \\ 0 \end{pmatrix} u + \begin{pmatrix} 0 \\ 0 \\ \sqrt{k_0} \end{pmatrix} e$$
$$z = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} x, \quad \Phi_e(\omega) = 1$$

3.5 a. With $\{A, B, C, N\}$ according to the solution of Problem 3.4, we have

$$\dot{x} = Ax + Bu + Ne$$
$$y = Cx + n$$

where n has spectral density $\Phi_n = 0.1$.

b. A noise signal with the specified spectral density is given by the output of a linear system with white noise input that has spectral density $\Phi_{w_n} = 0.1$. The transfer function of the system is

$$G_n(s) = \frac{s}{s+b} = \frac{s+b-b}{s+b} = 1 - \frac{b}{s+b}$$

In state-space form this can be expressed as

$$\dot{x}_4 = -bx_4 + bw_n$$
$$n = -x_4 + w_n$$

Combining the noise model with our original system gives the expanded state-space description:

$$\dot{x} = \begin{pmatrix} A & 0 \\ 0 & -b \end{pmatrix} x + \begin{pmatrix} B \\ 0 \end{pmatrix} u + \begin{pmatrix} N & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} e \\ w_n \end{pmatrix}$$
$$y = \begin{pmatrix} C & -1 \end{pmatrix} x + w_n, \quad \Phi_{\omega_n} = 0.1$$

Note that the disturbance can be described using a transfer function and white noise of any spectral density. For instance, it is often convenient to assume white noise with a spectral density of 1. In this case, the transfer function of the system would be

$$G_n(s) = \frac{\sqrt{0.1}s}{s+b}$$

The expanded state space description would then need to be adjusted to account for this.

c. Now, the transfer function of the noise model is $G_n(s) = \frac{1}{s+b}$. In state-space form, this is

$$\dot{x}_4 + bx_4 = w_n.$$

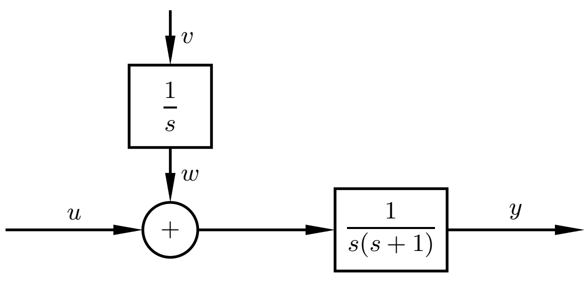
The expanded system becomes

$$\dot{x} = \begin{pmatrix} A & 0 \\ 0 & -b \end{pmatrix} x + \begin{pmatrix} B \\ 0 \end{pmatrix} u + \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e \\ w_n \end{pmatrix}$$
$$y = (C \quad 1)\, x, \quad \Phi_{\omega_n} = 0.1$$

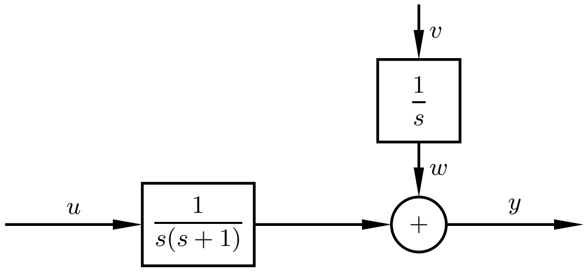
As in subproblem b, the disturbance can be described using a transfer function and white noise of any spectral density. Assuming white noise with a spectral density of 1, the transfer function of the system would be

$$G_n(s) = \frac{\sqrt{0.1}}{s+b}$$

3.6 a. (i)



(ii)



$v(t)$ is a unit disturbance

b. (i)

$$\dot{x} = \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}}_{A} x + \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{B} u + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} v$$
$$y = \underbrace{\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}}_{C} x.$$

(ii)

$$\dot{x} = \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{A} x + \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{B} u + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} v$$
$$y = \underbrace{\begin{pmatrix} 1 & 0 & 1 \end{pmatrix}}_{C} x.$$

c. (i) $w(t)$ could be an offset current on the input to the motor, and/or a step disturbance in the load.

(ii) In this case $w(t)$ could be a measurement disturbance, i.e., an additive error (constant) in the angle measurement. It could also be interpreted as a load disturbance on the process output. A controller could remove the effect from a load disturbance on the process output, but not a constant measurement disturbance, so the interpretation makes a difference.