FRTN10 Exercise 5. Controllability/Observability, Multivariable Poles and Zeros, Minimal Realizations

5.1 A TITO system has the state-space description

$$\dot{x} = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} x$$

a. Show that the system is controllable but not observable.

b. Give an example of a silent state, i.e., an initial state $x_0 \neq \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$ that will generate a constant output $y(t) = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$ when $u(t) = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$.

c. Calculate the minimum input energy, $\int_0^\infty |u(t)|^2 dt$, required to reach the state $x = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ from the origin.

5.2 Consider the following system, which is given in diagonal form:

$$\dot{x} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix} x$$

a. Show that the system is neither controllable nor observable by finding the uncontrollable and unobservable states. (This can be done by inspection.) Draw a block diagram that illustrates the situation.

b. Determine the transfer function of the system and the order of a minimal state-space realization. How can this be related to the controllable and observable states of the system?

5.3 The following is a model of a heat exchanger:

$$\dot{x} = \begin{pmatrix} -0.21 & 0.2 \\ 0.2 & -0.21 \end{pmatrix} x + \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x$$

Here the first state represents the temperature of the cold water and the second state is the temperature of the warm water.

a. State a set of linear equations (a Lyapunov equation) for finding the controllability Gramian W_c.

b. Solving the above Lyapunov equation gives

$$W_c = \begin{bmatrix} 0.256 & 0.244 \\ 0.244 & 0.256 \end{bmatrix}$$

with eigenvalues $\lambda_1 = 0.00122$ and $\lambda_2 = 0.5$ and corresponding eigenvectors

$$v_1 = \begin{bmatrix} -0.707 \\ 0.707 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix}$$

Is the system controllable? What state direction is the hardest to control?

5.4 In Figure 5.1 and Figure 5.2 you see two different interconnections of the two systems

$$P_1 = \frac{s+3}{s+2}, \quad P_2 = \frac{s+1}{(s+3)(s+4)(s-2)}$$

One can notice that after multiplying the two systems we can cancel a pole and a zero in $p_0 = -3$. Usually it means that the whole system is not observable, or not controllable. Which of these two situations are depicted in the systems A and B in Figure 5.1 and Figure 5.2 respectively?

Figure 5.1 Block diagram for system A in Problem 5.4.

Figure 5.2 Block diagram for system B in Problem 5.4.

5.5 Consider the following transfer function matrix

$$G(s) = \begin{pmatrix} \dfrac{1}{s+2} & -\dfrac{1}{s+2} \\ \dfrac{1}{s+2} & \dfrac{s+1}{s+2} \end{pmatrix}$$

a. Determine the pole and zero polynomials for this system. What is the least order needed to realize the system in state-space form?

b. Find a state-space realization of the system.

5.6 Consider the following 2×3 system (with three inputs and two outputs):

$$G(s) = \frac{1}{(s+1)(s+2)(s-1)} \begin{pmatrix} (s-1)(s+2) & 0 & (s-1)^2 \\ -(s+1)(s+2) & (s-1)(s+1) & (s-1)(s+1) \end{pmatrix}$$

Determine the poles and zeros of the system.

5.7 **(*)** Consider the system

$$G(s) = \begin{pmatrix} 1 & 1/s \end{pmatrix}$$

with two inputs and one output.

a. Use Matlab to determine the singular values of the system at $\omega = 1$ rad/s, together with the input directions giving the maximum and minimum output gains respectively.

b. The derived input directions are complex. What does this mean? Explain why it is logical that these input directions should give the smallest and highest system gains respectively for this particular system.

5.8 (*) The following is a rough model of the pitch dynamics of JAS 39 Gripen:

$$\dot{x} = \begin{pmatrix} -1 & 1 & 0 & -1/2 & 0 \\ 4 & -1 & 0 & -25 & 8 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -20 & 0 \\ 0 & 0 & 0 & 0 & -20 \end{pmatrix} x + \begin{pmatrix} 0 & 0 \\ 3/2 & 1/2 \\ 0 & 0 \\ 20 & 0 \\ 0 & 20 \end{pmatrix} u.$$

Using Matlab:

a. Show that there is no scalar output signal that makes the system observable.

Hint: Use symbolic toolbox to determine a general C matrix and calculate the observability matrix $\mathcal{O}$. For instance, the following lines of Matlab code may help you:

```
>> syms c1 c2 c3 c4 c5
>> C = [c1 c2 c3 c4 c5]
>> O = ...
```

b. Let the output be

$$y(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} x(t).$$

Which are the non-observable modes?

Solutions to Exercise 5. Controllability/Observability, Multivariable Poles and Zeros, Minimal Realizations

5.1 **a.** The controllability matrix $\mathcal{C} = [B \quad AB] = \left[\begin{smallmatrix} 1 & -1 & -3 & 3 \\ 0 & 0 & 1 & -1 \end{smallmatrix}\right]$ has full rank (2), meaning that the system is controllable.

The observability matrix $\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \left[\begin{smallmatrix} 1 & 1 \\ -1 & -1 \\ -2 & -2 \\ 2 & 2 \end{smallmatrix}\right]$ has rank 1, meaning that the system is not observable.

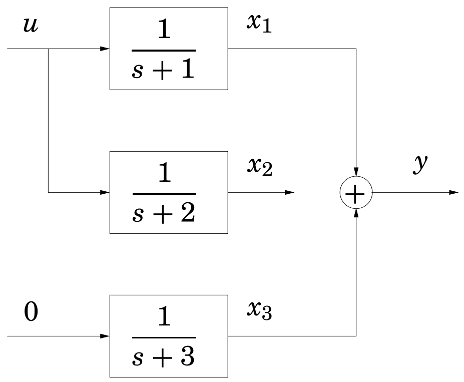
b. The silent states are given by the null space of the observability matrix, i.e., by $\mathcal{O}x_0 = 0$. All solutions are given by $x_0 = \left[\begin{smallmatrix} t \\ -t \end{smallmatrix}\right]$, so for instance $x_0 = \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]$ works.

c. The controllability Gramian W_c is given by the Lyapunov equation

$$AW_c + W_cA^T + BB^T = 0$$

with the solution $W_c = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{6} \end{bmatrix}$. The minimum energy required to reach $x_1 = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ is given by $x_1^T W_c^{-1} x_1 = 9$.

5.2 **a.** Since the system is written in diagonal form, we can directly see that x_3 cannot be influenced by the control signal and that x_2 does not influence the measurement signal. The situation is illustrated in the block diagram below.



b. The transfer function from u to y is simply

$$G(s) = C(sI - A)^{-1}B = \frac{1}{s+1}$$

and the system can thus be represented as a minimal state-space realization of order 1. Note that this corresponds to the first subsystem above, which is both observable and controllable.

If we are only interested in the relationship between u and y, we can use the resulting first-order transfer function $G(s)$. However, the original third-order state-space model contains additional information, as seen in the block diagram above. The second and third subsystems in this model may represent physical entities of the plant that must be taken into account. If we need to influence x_3 or monitor x_2, additional sensors or actuators are needed.

5.3 **a.** The controllability Gramian $W_c = \begin{bmatrix} w_{c1} & w_{c2} \\ w_{c2} & w_{c3} \end{bmatrix}$ is given by the solution to

$$AW_c + W_c A^T + BB^T = 0$$

yielding the linear equations

$$\begin{aligned} -0.42w_{c1} + 0.4w_{c2} + 0.01 &= 0 \\ 0.2w_{c1} - 0.42w_{c2} + 0.2w_{c3} &= 0 \\ 0.4w_{c2} - 0.42w_{c3} + 0.01 &= 0 \end{aligned}$$

b. The system is controllable, since W_c has full rank. It is however difficult to control the system in the direction v_1 corresponding to the near-zero eigenvalue. The interpretation is that it is difficult to achieve very different temperatures x_1 and x_2.

5.4 System A depicts the observable but not controllable system. Obviously the problem is in the pole $p_0 = -3$. We control directly the plant P_1, and we observe the output of plant P_2. It means that we observe the effect of the pole $p_0 = -3$, but due to pole-zero cancellation, we cannot control it.

Similarly for system B, we control the plant P_2, and the pole $p_0 = -3$ is controllable, but the effect of that pole is cancelled by the zero in P_1 and we do not observe it. Hence the whole system is not observable.

5.5 **a.** The largest subdeterminant of the transfer function matrix is

$$\frac{(s+1)}{(s+2)^2} + \frac{1}{(s+2)^2} = \frac{1}{(s+2)}$$

Furthermore, the matrix elements in themselves are subdeterminants. The pole polynomial, i.e. the least common denominator of all subdeterminants, is then

$$p(s) = (s+2)$$

This means that the system has a pole in $s = -2$. The system can thus be realized in state-space form of order 1.

The largest possible subdeterminant was

$$\frac{1}{(s+2)}$$

The zero polynomial is thus just a constant and therefore the system does not have any zeros.

b.

$$G(s) = \begin{pmatrix} \frac{1}{s+2} & -\frac{1}{s+2} \\ \frac{1}{s+2} & \frac{s+1}{s+2} \end{pmatrix} = \begin{pmatrix} \frac{1}{s+2} & -\frac{1}{s+2} \\ \frac{1}{s+2} & 1 - \frac{1}{s+2} \end{pmatrix} = \frac{1}{s+2}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$
$$= \frac{1}{s+2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

A state-space realization can now be written as

$$\frac{dx}{dt} = -2x + \begin{pmatrix} 1 & -1 \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 \\ 1 \end{pmatrix} x + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} u$$

5.6 The relevant subdeterminants of order 1 are the five non-zero elements

$$\frac{1}{s+1}, \quad \frac{s-1}{(s+1)(s+2)}, \quad \frac{-1}{(s-1)}, \quad \frac{1}{(s+2)}, \quad \frac{1}{(s+2)}$$

and the 3 subdeterminants of order 2, corresponding to deletion of the columns, are

$$\frac{-(s-1)}{(s+1)(s+2)^2}, \quad \frac{2}{(s+1)(s+2)}, \quad \frac{1}{(s+1)(s+2)}.$$

Considering all subdeterminants, we see that the least common denominator is

$$p(s) = (s+1)(s+2)^2(s-1).$$

The system has therefore four poles: one at $s = -1$, one at $s = 1$ and two at $s = -2$.

To determine the zeros of the system, adjust the subdeterminants of order two so that their denominators are the pole polynomial $p(s)$. We get

$$\frac{-(s-1)^2}{p(s)}, \quad \frac{2(s-1)(s+2)}{p(s)}, \quad \frac{(s-1)(s+2)}{p(s)}.$$

The common factor for these subdeterminants is the zero polynomial $z(s) = (s-1)$. Thus, the system has a single RHP-zero located at $s = 1$.

5.7 a. To determine the frequency response at a certain frequency ω, it's handy to use the Matlab command freqresp. To calculate the singular values together with the U and V matrices, use the function svd. The Matlab code can look like this:

```
>> s = tf('s');
>> G = [1 1/s];
>> [U,S,V] = svd(freqresp(G,1))
U =
     1
S =
    1.4142         0
V =
   0.7071        0 + 0.7071i
   0 + 0.7071i   0.7071
```

The maximum gain, corresponding to the highest singular value, is obtained as the first element in S and is $\overline{\sigma} = 1.4142$. The first column of V, $v_1 = (0.7071 \quad 0.7071i)^T$, corresponds to the input direction that gives the maximum gain $\overline{\sigma}$. Since the system has two inputs and only one output, there will always be an input direction that gives zero output (where the inputs cancel each other). The second column of V gives us this direction, $v_2 = (0.7071i \quad 07071)^T$.

b. If the input signal is a sinusoid with frequency $\omega = 1$ rad/s, angle between the complex numbers will correspond to a phase shift of this sinusoid. The input direction giving the highest gain is $v_1 = [0.7071 \quad 0.7071i]^T$, meaning that the second input has 90° phase lead compared to the the first input.

The first input comes through the system unchanged; the second goes through an integrator, causing a phase lag of 90°. Thus the input direction $v_1 = [0.7071 \quad 0.7071i]^T$ will cause the two sinusoids that sum up at the output to be in phase; resulting in maximal gain.

If we instead use the lowest gain input direction $v_2 = [0.7071i \quad 0.7071]^T$, the second input will have a phase lag of 90°, causing a 180° phase lag at the output. The two signals will cancel at the output, resulting in zero gain.

5.8 a. Continuing the code we get

```
>> syms c1 c2 c3 c4 c5
>> C = [c1 c2 c3 c4 c5];
>> O = [C;C*A;C*A^2;C*A^3;C*A^4];
>> det(O)
ans =
0
>> rank(O)
ans =
4
```

Since the system does not have full rank (5) we see that no matter how we choose C (when it is a vector), the system can never be made observable. This means that we need information from more than just one signal to make the system observable.

b. Determine the eigenvectors of the system

```
>> [V,D]=eig(A)
V =
0      0.3333     0.4286    -0.0261     0.0206
0      0.6667    -0.8571     0.7973    -0.3916
1.0000     0.6667     0.2857    -0.0399     0.0196
0          0          0     0.6017          0
0          0          0          0     0.9197
...
```

Rewrite the system in diagonal form using the change of variables $x(t) = Vz(t)$:

$$
\begin{aligned}
\dot{x}(t) &= V\dot{z}(t) = AVz(t) + Bu(t) \Rightarrow \\
\dot{z}(t) &= V^{-1}AVz(t) + V^{-1}Bu(t) = \Lambda z(t) + V^{-1}Bu(t) \\
y(t) &= CVz(t)
\end{aligned}
$$

where Λ is a diagonal matrix with the eigenvalues on the diagonal. Now that we have the system in the wanted form, we can determine if there are any columns in CV that are zero.

```
>> C*V
ans =
0    0.3333    0.4286   -0.0261    0.0206
0    0.6667   -0.8571    0.7973   -0.3916
```

The first state in z therefore corresponds to the unobservable mode. In the original variables this is the third state:

```
>> V*[1;0;0;0;0]
ans =
0
0
1
0
0
```

So, the third state is the unobservable mode.