FRTN10 Exercise 7. Decentralized Control, Preparations for Lab 2

Note: Exercise 7.2 serves as preparation for Laboratory Excercise 2. You are expected to be able to discuss around the discussion points in the lab manual and to show the calculations leading to the suggested input-output pairing in Problem 7.2**d.**

7.1 A MIMO process is described by the transfer matrix

$$P(s) = \begin{pmatrix} \frac{1}{s+2} & \frac{10}{s+1} \\ \frac{1}{s+5} & \frac{5}{s+3} \end{pmatrix}.$$

a. Compute RGA$(P(0))$. What input-output pairing would you recommend in a decentralised control structure?

b. Assume that two identical PI controllers with gain $K = 1$ and integral time $T_i = 0.2$ are used to control the two process outputs. The possible pairings investigated above then correspond to the two controllers

$$C_1(s) = \begin{pmatrix} \frac{s+5}{s} & 0 \\ 0 & \frac{s+5}{s} \end{pmatrix}, \qquad C_2(s) = \begin{pmatrix} 0 & \frac{s+5}{s} \\ \frac{s+5}{s} & 0 \end{pmatrix}.$$

For each controller C_i, $i = \{1, 2\}$, compute the closed-loop system

$$T = PC_i\,(I + PC_i)^{-1}$$

using the command `feedback` and plot its step responses using `step`. Which configuration works the best? Do the results agree with the RGA analysis?

7.2 Figure 7.1 shows the quadruple-tank process that will be used in Lab 2. The goal is to control the measured levels in the lower tanks (y_1, y_2) using the pumps (u_1, u_2). For each tank $i = 1 \ldots 4$, mass balance and Torricelli's law give that

$$A_i \frac{dh_i}{dt} = -a_i\sqrt{2gh_i} + q_{in} \tag{7.1}$$

where A_i is the cross-section of the tank, h_i is the water level, a_i is the cross-section of the outlet hole, g is the acceleration of gravity, and q_{in} is the inflow to the tank. The non-linear equation (7.1) can be linearized around a stationary point (h_i^0, q_{in}^0), giving the linear equation

$$A_i \frac{d\Delta h_i}{dt} = -a_i\sqrt{\frac{g}{2h_i^0}}\,\Delta h_i + \Delta q_{in} \tag{7.2}$$

where $\Delta h_i = h_i - h_i^0$, and $\Delta q_{in} = q_{in} - q_{in}^0$ denote deviations around the stationary point.

The flows from the pumps are divided according to two parameters $\gamma_1, \gamma_2 \in (0, 1)$. The flow into Tank 1 is $\gamma_1 k_1 u_1$ and the flow into Tank 4 is $(1-\gamma_1)k_1 u_1$.

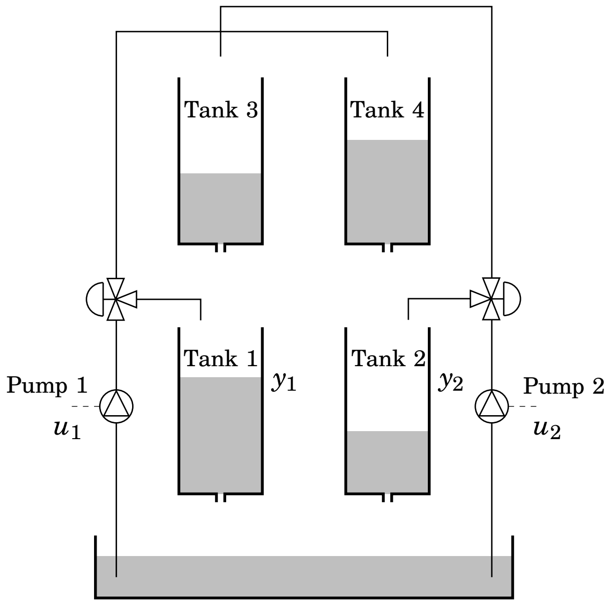

Figure 7.1 The quadruple-tank process.

Symmetrically, the flow into Tank 2 is $\gamma_2 k_2 u_2$ and the flow into Tank 3 is $(1-\gamma_2)k_2u_2$.

The measurement signals are given by $y_1 = k_c h_1$ and $y_2 = k_c h_2$, where k_c is a measurement constant.

a. Let $\Delta u_i = u_i - u_i^0$, $\Delta h_i = h_i - h_i^0$, and $\Delta y_i = y_i - y_i^0$. Verify that the linearized dynamics of the complete quadruple-tank system is given by

$$\begin{aligned}
\frac{d\Delta h_1}{dt} &= -\frac{a_1}{A_1}\sqrt{\frac{g}{2h_1^0}}\,\Delta h_1 + \frac{a_3}{A_1}\sqrt{\frac{g}{2h_3^0}}\,\Delta h_3 + \frac{\gamma_1 k_1}{A_1}\,\Delta u_1 \\
\frac{d\Delta h_2}{dt} &= -\frac{a_2}{A_2}\sqrt{\frac{g}{2h_2^0}}\,\Delta h_2 + \frac{a_4}{A_2}\sqrt{\frac{g}{2h_4^0}}\,\Delta h_4 + \frac{\gamma_2 k_2}{A_2}\,\Delta u_2 \\
\frac{d\Delta h_3}{dt} &= -\frac{a_3}{A_3}\sqrt{\frac{g}{2h_3^0}}\,\Delta h_3 + \frac{(1-\gamma_2)k_2}{A_3}\,\Delta u_2 \\
\frac{d\Delta h_4}{dt} &= -\frac{a_4}{A_4}\sqrt{\frac{g}{2h_4^0}}\,\Delta h_4 + \frac{(1-\gamma_1)k_1}{A_4}\,\Delta u_1
\end{aligned}$$

Introduce the input vector, u, state vector, x, and output vector, y, as

$$u = \begin{pmatrix} \Delta u_1 \\ \Delta u_2 \end{pmatrix}, \quad x = \begin{pmatrix} \Delta h_1 \\ \Delta h_2 \\ \Delta h_3 \\ \Delta h_4 \end{pmatrix}, \quad y = \begin{pmatrix} \Delta y_1 \\ \Delta y_2 \end{pmatrix}.$$

Verify that the linearized system can be written in state-space form as

$$\frac{dx}{dt} = \begin{pmatrix} -\frac{1}{T_1} & 0 & \frac{A_3}{A_1T_3} & 0 \\ 0 & -\frac{1}{T_2} & 0 & \frac{A_4}{A_2T_4} \\ 0 & 0 & -\frac{1}{T_3} & 0 \\ 0 & 0 & 0 & -\frac{1}{T_4} \end{pmatrix} x + \begin{pmatrix} \frac{\gamma_1 k_1}{A_1} & 0 \\ 0 & \frac{\gamma_2 k_2}{A_2} \\ 0 & \frac{(1-\gamma_2)k_2}{A_3} \\ \frac{(1-\gamma_1)k_1}{A_4} & 0 \end{pmatrix} u,$$

$$y = \begin{pmatrix} k_c & 0 & 0 & 0 \\ 0 & k_c & 0 & 0 \end{pmatrix} x,$$

where $T_i = \frac{A_i}{a_i}\sqrt{\frac{2h_i^0}{g}}$.

b. Show that the transfer matrix from u to y is given by

$$P(s) = \begin{pmatrix} \frac{\gamma_1 c_1}{1+sT_1} & \frac{k_2}{k_1}\cdot\frac{(1-\gamma_2)c_1}{(1+sT_1)(1+sT_3)} \\ \frac{k_1}{k_2}\cdot\frac{(1-\gamma_1)c_2}{(1+sT_2)(1+sT_4)} & \frac{\gamma_2 c_2}{1+sT_2} \end{pmatrix}$$

where $c_1 = T_1k_1k_c/A_1$ and $c_2 = T_2k_2k_c/A_2$.
Hint: Use the fact that

$$\begin{pmatrix} a & 0 & b & 0 \\ 0 & c & 0 & d \\ 0 & 0 & e & 0 \\ 0 & 0 & 0 & f \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{a} & 0 & -\frac{b}{ae} & 0 \\ 0 & \frac{1}{c} & 0 & -\frac{d}{cf} \\ 0 & 0 & \frac{1}{e} & 0 \\ 0 & 0 & 0 & \frac{1}{f} \end{pmatrix}$$

c. The zeros are given by the equation

$$\det P(s) = \frac{c_1c_2\big(\gamma_1\gamma_2(1+sT_3)(1+sT_4) - (1-\gamma_1)(1-\gamma_2)\big)}{(1+sT_1)(1+sT_2)(1+sT_3)(1+sT_4)} = 0$$

which can be simplified to

$$(1+sT_3)(1+sT_4) - \frac{(1-\gamma_1)(1-\gamma_2)}{\gamma_1\gamma_2} = 0.$$

Show that the system is minimum phase (i.e., that both zeros are in the left half-plane) if $1 < \gamma_1 + \gamma_2 < 2$, and that the system is non-minimum phase (i.e., that at least one zero is in the right half-plane) if $0 < \gamma_1 + \gamma_2 < 1$. Remember that $\gamma_1, \gamma_2 \geq 0$.

Hint: A second-order polynomial has all of its roots in the left half-plane if and only if all coefficients have the same sign.

In the lab, we will first study the case $\gamma_1 = \gamma_2 \approx 0.7$, and then the case $\gamma_1 = \gamma_2 \approx 0.3$. In which case will the process be more difficult to control?

d. Show that the RGA for $P(0)$ is given by

$$\begin{pmatrix} \lambda & 1-\lambda \\ 1-\lambda & \lambda \end{pmatrix}$$

where $\lambda = \gamma_1\gamma_2/(\gamma_1+\gamma_2-1)$.
Based on this RGA matrix, suggest an input-output pairing in the two cases $\gamma_1 = \gamma_2 \approx 0.7$ and $\gamma_1 = \gamma_2 \approx 0.3$.

7.3 Consider the MIMO process

$$P(s) = \begin{pmatrix} \dfrac{1}{s+1} & 0 & 0 \\ 0 & \dfrac{0.1}{s+10} & \dfrac{1}{s+10} \\ \dfrac{0.1}{s+1} & \dfrac{1}{s+1} & 0 \end{pmatrix}.$$

Compute the relative gain array, RGA, of $P(0)$ and suggest an input-output pairing for the system based on this.
Hint: The inverse of $P(s)$ is given by

$$P(s)^{-1} = \begin{pmatrix} s+1 & 0 & 0 \\ -0.1(s+1) & 0 & s+1 \\ 0.01(s+1) & s+10 & -0.1(s+1) \end{pmatrix}.$$

7.4 Consider the following multivariable system

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \dfrac{1}{10s+1} & \dfrac{-2}{2s+1} \\ \dfrac{1}{10s+1} & \dfrac{s-1}{2s+1} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

a. By using RGA at stationarity, decide the input-output pairing that should be used in a decentralized control structure.

b. Assume that we want to use decentralized control, that is, we want to use a controller that can be described by

$$C^{\text{diag}}(s) = \begin{pmatrix} C_{11}(s) & 0 \\ 0 & C_{22}(s) \end{pmatrix}.$$

Also, we want the control loops to be decoupled in stationarity. Give the structure of such a controller $C(s)$ expressed in $C^{\text{diag}}(s)$ that is capable to do so. *Hint: Use a suitable decoupling matrix.*

7.5 (*) In this exercise we will try to design controllers for a 2×2-process, that is, a process that has 2 inputs and 2 outputs. The process is described by the transfer function matrix

$$G(s) = \begin{pmatrix} \dfrac{4}{s+1} & \dfrac{3}{3s+1} \\ \dfrac{1}{3s+1} & \dfrac{2}{s+0.5} \end{pmatrix}.$$

Design two different decentralized controllers for the process.

1. Decentralized control, using the RGA of the process.
2. Decentralized control, using decoupling with respect to stationarity

In both cases, use ordinary PI controllers. Use the step responses to evaluate the performance of the loop.

Solutions to Exercise 7. Decentralized Control, Preparations for Lab 2

7.1 **a.**

$$\mathrm{RGA}(P(0)) = P(0).* P^{-T}(0) = \begin{pmatrix} -\frac{5}{7} & \frac{12}{7} \\ \frac{12}{7} & -\frac{5}{7} \end{pmatrix}$$

Since we should avoid negative relative gains we should choose the pairing $y_1 \leftrightarrow u_2$ and $y_2 \leftrightarrow u_1$.

b.

```
>> s = zpk('s');
>> P = [1/(s+2) 10/(s+1); 1/(s+5) 5/(s+3)];
>> C1 = [(s+5)/5 0; 0 (s+5)/s];
>> T1 = feedback(P*C1,eye(2));
>> step(T1)
>> C2 = [0 (s+5)/5; (s+5)/s 0];
>> T2 = feedback(P*C2,eye(2));
>> step(T2)
```

The responses are seen in Figures 7.1 and 7.2. It is seen that, with $C_1(s)$, the closed-loop system is unstable, while $C_2(s)$ produces a relatively well-behaved stable closed-loop system. This agrees with the RGA analysis, which suggested that the controller C_1 should be avoided.

7.2 *No solution provided.*

7.3 We have

$$P(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.01 & 0.1 \\ 0.1 & 1 & 0 \end{pmatrix}$$

and

$$P(0)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -0.1 & 0 & 1 \\ 0.01 & 10 & -0.1 \end{pmatrix}$$

$$\mathrm{RGA}(P(0)) = P(0).* (P(0)^{-1})^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

The RGA suggests that we should control output 1 with input 1, output 2 with input 3, and output 3 with input 2.

7.4 **a.** We compute the RGA for stationarity, i.e. $s = 0$.

$$\mathrm{RGA}(G(s)) = \begin{pmatrix} \frac{s-1}{s+1} & \frac{2}{s+1} \\ \frac{2}{s+1} & \frac{s-1}{s+1} \end{pmatrix}$$

gives

$$\mathrm{RGA}(G(0)) = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}.$$

Since you should avoid pairing that gives negative diagonal elements we choose $y_1 \leftrightarrow u_2$ and $y_2 \leftrightarrow u_1$.

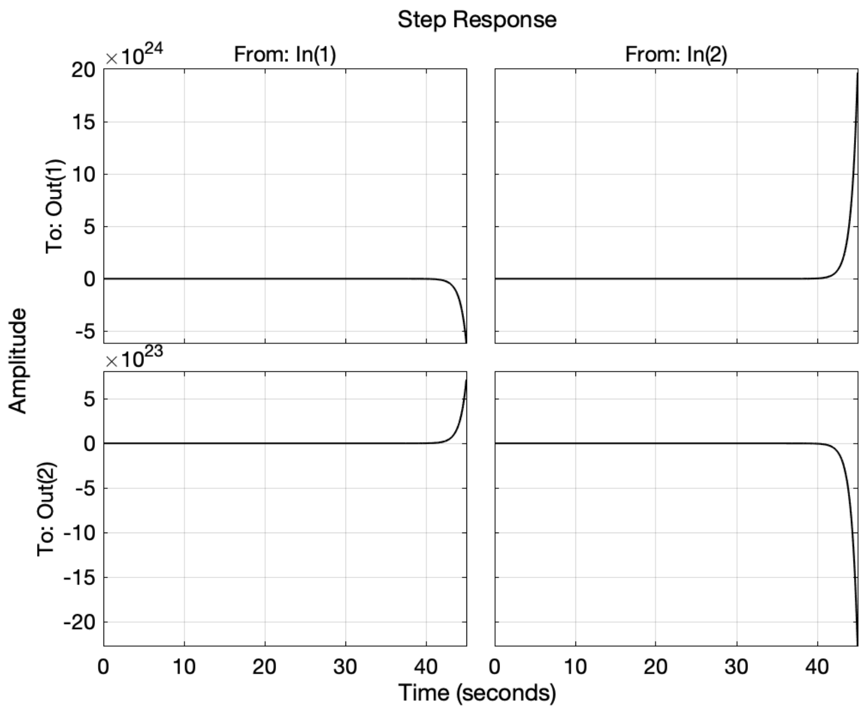

Figure 7.1 Closed-loop system with controller C_1 in Problem 7.1

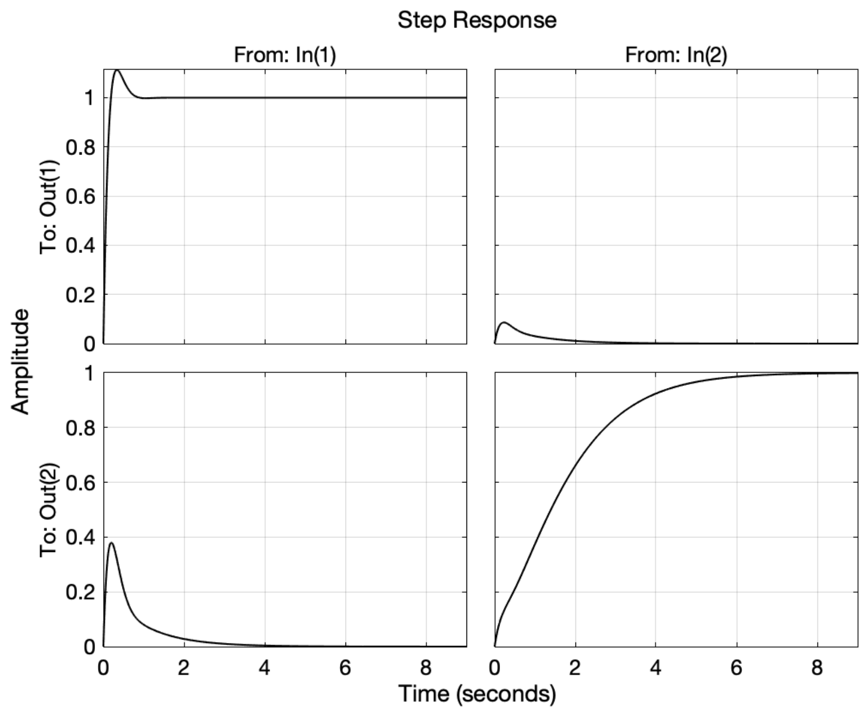

Figure 7.2 Closed-loop system with controller C_2 in Problem 7.1

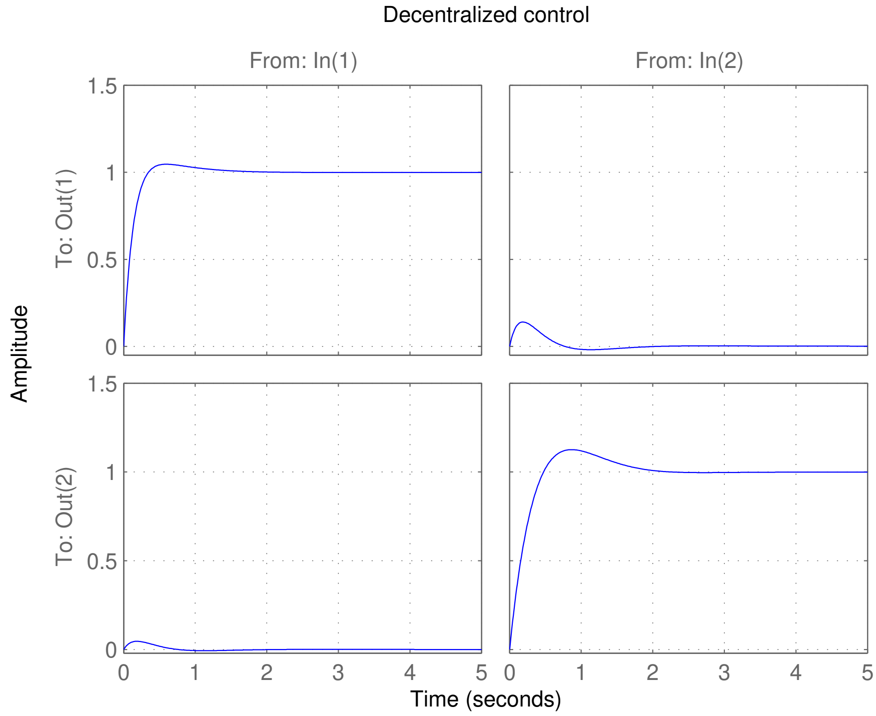

Figure 7.3 Decentralized control

b. We have that

$$G(0) = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}$$

Using a decoupled controller structure with $W_1 = G^{-1}(0)$ and $W_2 = I$ we get a decoupled system in stationarity. The controller is

$$C(s) = W_1 F^{\text{diag}}(s) W_2 = \begin{pmatrix} -C_{11}(s) & 2C_{22}(s) \\ -C_{11}(s) & C_{22}(s) \end{pmatrix}.$$

7.5

1. Decentralized control. First we calculate the RGA of the process,

$$\text{RGA}(G(0)) = G(0).*G^{-T}(0) = \begin{pmatrix} 1.2308 & -0.2308 \\ -0.2308 & 1.2308 \end{pmatrix}.$$

We see that we should choose $y_1 \leftrightarrow u_1$ and $y_2 \leftrightarrow u_2$. A resonable tuning, either by pole placement or hand tuning, gives PI controllers with parameters close to

$$F(s) = \begin{pmatrix} 2(1 + \frac{1}{0.5s}) & 0 \\ 0 & 2(1 + \frac{1}{0.5s}) \end{pmatrix}.$$

See figure 7.3 for step responses.

2. Decoupled control. The inverse of the static gain matrix is given by

$$G^{-1}(0) = \begin{pmatrix} 4 & 3 \\ 1 & 4 \end{pmatrix}^{-1}$$

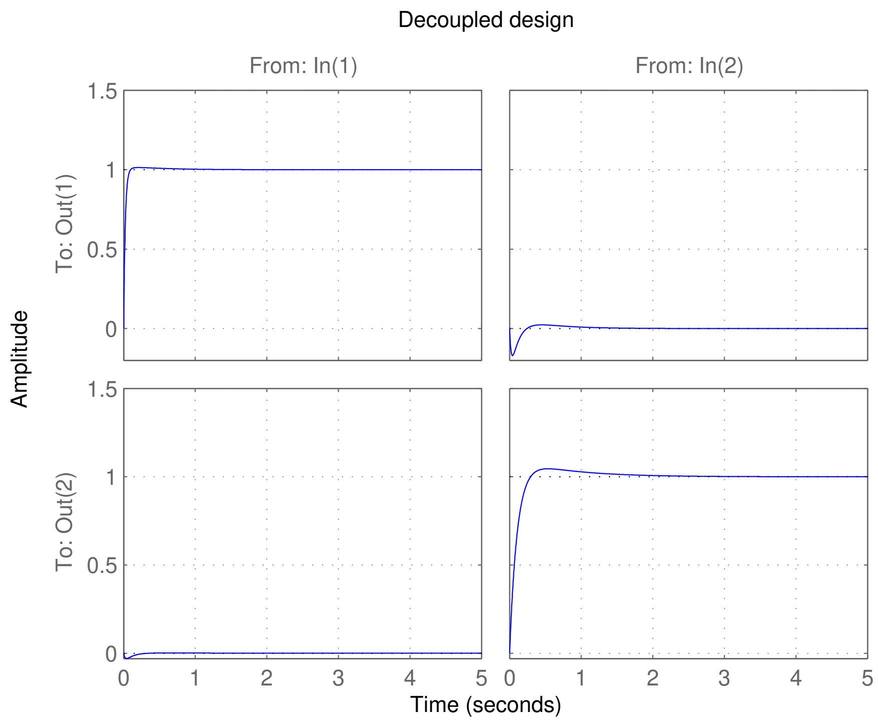

Figure 7.4 Decoupled control

Thus, for decoupling, we use $W_1 = G^{-1}(0)$ and $W_2 = I$. Hand-tuning of the PI controllers gives

$$F(s) = \begin{pmatrix} 40(1 + \frac{1}{0.5s}) & 0 \\ 0 & 20(1 + \frac{1}{0.8s}) \end{pmatrix}.$$

See figure 7.4 for step responses.

Matlab code:

```
s = zpk('s');
G = [4/(s+1) 3/(3*s+1); 1/(3*s+1) 2/(s+0.5)];

% Decentralized control
RGA = dcgain(G).*(inv(dcgain(G))).'
F = [2*(1+1/(0.5*s)) 0;0 2*(1+1/(0.5*s))];
figure(1)
step(feedback(G*F, eye(2)),5)
title('Decentralized control');grid

% Decoupled design
Go = dcgain(G)
F = [40*(1+1/(0.5*s)) 0;0 20*(1+1/(0.8*s))];
figure(2);
step(feedback(G*inv(Go)*F,eye(2)),5);
title('Decoupled design');grid
```