FRTN10 Exercise 8. Linear-Quadratic Control

8.1 Consider the first-order unstable process

$$\dot{x}(t) = ax(t) + u(t), \qquad a > 0$$

a. Design an LQ controller $u(t) = -Lx(t)$ that minimizes the criterion

$$J = \int_0^\infty \left(x^2(t) + \rho u^2(t)\right) dt, \qquad \rho > 0$$

b. Calculate the location of the closed-loop as a function of ρ and discuss what happens when either $\rho \to 0$ or $\rho \to \infty$.

8.2 Consider the second-order system

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} u(t) \\ y(t) &= \begin{pmatrix} 5 & 5 \end{pmatrix} x(t) \end{aligned}$$

Design an LQ controller $u(t) = -Lx(t)$ that minimizes the criterion

$$J = \int_0^\infty \left(y^2(t) + 5u_1^2(t) + 5u_2^2(t)\right) dt.$$

What are the poles of the closed-loop system? Also calculate the minimal value of J when the initial state is $x(0) = \left(\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right)$.

8.3 Consider a process

$$\dot{x}(t) = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} x(t) + \begin{pmatrix} 3 \\ 2 \end{pmatrix} u(t)$$

Show that $u(t) = -Lx(t)$ with

$$L = \begin{pmatrix} 2 & -3 \end{pmatrix}$$

can *not* be an optimal state feedback designed using linear-quadratic control theory with the cost function

$$J = \int_0^\infty \left(x^T(t)Q_1x(t) + Q_2u^2(t)\right) dt$$

where $Q_1, Q_2 > 0$.

Hint: Sketch the Nyquist plot of the loop transfer function $L(sI - A)^{-1}B$.

8.4 Consider the system

$$\dot{x} = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix} x + \begin{pmatrix} -4 \\ 8 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 1 \end{pmatrix} x$$

One wishes to minimize the criterion

$$J(T) = \int_0^T \left(x^T(t) Q_1 x(t) + Q_2 u^2(t) \right) dt$$

Is it possible to find positive definite weights Q_1 and Q_2 such that the cost function $J(T) < \infty$ as $T \to \infty$?

8.5 Consider the double integrator

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t)$$
$$y(t) = \begin{pmatrix} 1 & 0 \end{pmatrix} x(t)$$

A set of LQ controllers $u(t) = -Lx(r) + L_r r(r)$ have been designed. L was calculated to minimize the cost function

$$J = \int_0^\infty \left(x^T(t) Q_1 x(t) + Q_2 u^2(t) \right) dt$$

and L_r was chosen to give unit static gain from r to y. The four plots in Figure 8.1 show the step responses of the closed-loop system for four different combinations of weights, Q_1, Q_2. Pair the combinations of weights given below with the step responses in Figure 8.1.

1)
$$Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Q_2 = 0.01$$

2)
$$Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Q_2 = 1$$

3)
$$Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Q_2 = 1$$

4)
$$Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Q_2 = 1000$$

8.6 (*) We would like to control the following process with linear-quadratic optimal control:

$$\dot{x}(t) = \begin{pmatrix} 1 & 3 \\ 4 & 8 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0.1 \end{pmatrix} u(t)$$
$$y(t) = \begin{pmatrix} 0 & 1 \end{pmatrix} x(t)$$

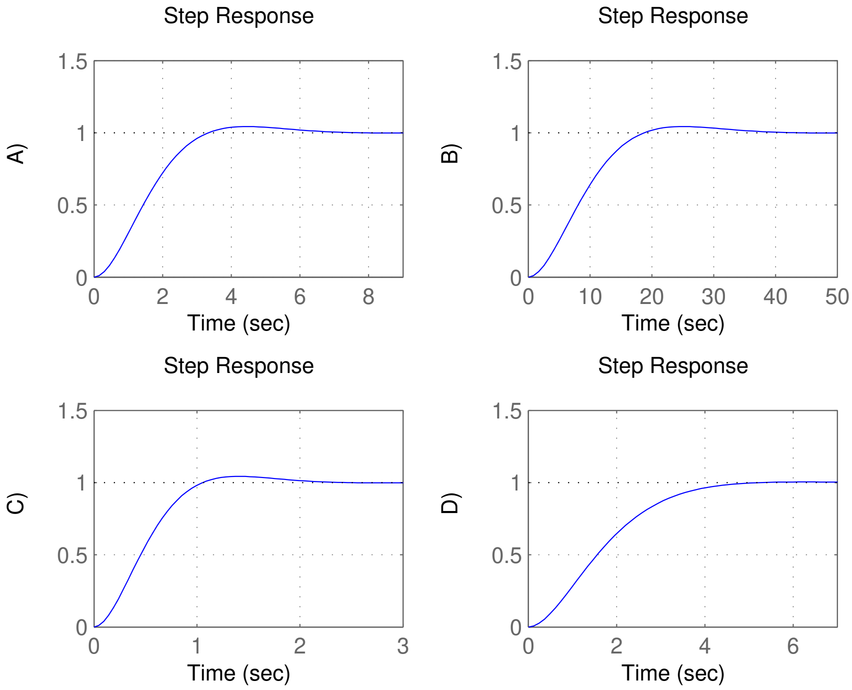

Figure 8.1 Step responses for LQ-control of the system in Problem 8.5 with different weights on Q_1, Q_2.

The penalty on $x_1^2(t)$ should be 1, and the penalty on $x_2^2(t)$ should be 2. For $u^2(t)$ we will try different penalty values: $\rho = 0.01,\ 1,\ 100$.

a. Determine the cost function for the three different cases.

b. Assume that we want to add reference tracking so that $y = r$ in stationarity, using the control law $u(t) = L_r r(t) - Lx(t)$. In Matlab, calculate the three different resulting controllers, calculate the resulting closed-loop poles and simulate step responses from r to x_2 and from r to u. Verify that there is no static error.

8.7 **(*)** Consider the double integrator

$$\ddot{\xi}(t) = u(t).$$

with state-space representation

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x$$

where $x = (\xi(t), \dot{\xi}(t))$. You would like to design a controller using the criterion

$$\int_0^\infty (\xi^2(t) + \eta \cdot u^2(t))\, dt$$

for some $\eta > 0$.

a. Show that $S = \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix}$ with

$$\begin{aligned} s_1 &= \sqrt{2} \cdot \eta^{1/4} \\ s_2 &= \eta^{1/2} \\ s_3 &= \sqrt{2} \cdot \eta^{3/4} \end{aligned}$$

solves the Riccati equation.

b. What are the closed-loop poles of the system when using this optimal state feedback? What happens with the control signal if η is reduced?

Solutions to Exercise 8. Linear-Quadratic Control

8.1 **a.** Using $A = a,\ B = 1,\ Q_1 = 1,\ Q_2 = \rho,\ Q_{12} = 0$ the Riccati equation becomes

$$2Sa + 1 - S\rho^{-1}S = 0$$

The positive solution is

$$S = a\rho + \sqrt{(a\rho)^2 + \rho}$$

and the optimal controller gain is given by

$$L = \frac{S}{\rho} = a + \sqrt{a^2 + \frac{1}{\rho}}.$$

b. The closed-loop pole is given by

$$A - BL = -\sqrt{a^2 + \frac{1}{\rho}}$$

The pole is located in the left half-plane for all $\rho > 0$. When $\rho \to 0$, control is cheap and the gain approaches ∞, while the pole approaches $-\infty$. When $\rho \to \infty$, control is expensive and the gain approaches $2a$, while the pole approaches $-a$ (actually the open-loop pole mirrored in the imaginary axis). It is interesting to note that, since the system is unstable, the gain cannot approach zero when control becomes expensive.

8.2 Using $y(t) = Cx(t)$ and $u(t) = \left(\begin{smallmatrix} u_1(t) \\ u_2(t) \end{smallmatrix}\right)$ we first rewrite the cost function as

$$J = \int_0^\infty \left(x^T(t)C^TCx(t) + u^T(t) \left(\begin{smallmatrix} 5 & 0 \\ 0 & 5 \end{smallmatrix}\right) u(t) \right) dt$$

from which we identify $Q_1 = C^TC = \left(\begin{smallmatrix} 25 & 25 \\ 25 & 25 \end{smallmatrix}\right)$, $Q_2 = \left(\begin{smallmatrix} 5 & 0 \\ 0 & 5 \end{smallmatrix}\right)$ and $Q_{12} = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$.
The Riccati equation is $Q_1 + A^TS + SA - SBQ_2^{-1}B^TS = 0$. Let

$$S = \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix}$$

We get the following system of equations:

$$\begin{aligned} -s_1^2 + 2s_1 + 2s_2 + 25 &= 0 \\ s_2 + s_3 - s_1s_2 + 25 &= 0 \\ 25 - s_2^2 &= 0 \end{aligned}$$

The positive solution is $s_1 = 7$, $s_2 = s_3 = 5$. This gives the state feedback gain

$$L = Q_2^{-1}B^TS = \begin{pmatrix} 2.8 & 2 \\ 1.4 & 1 \end{pmatrix}.$$

The poles of the closed-loop system are given by $\det(\lambda I - A + BL) = 0$ which gives $\lambda_1 = -1$, $\lambda_2 = -5$.
The minimal value of J is given by $x^T(0)Sx(0) = 2$.

8.3 The loop gain is

$$L(sI - A)^{-1}B = \frac{6}{(s+1)(s+2)}$$

The Nyquist curve starts on the positive real axis and will approach the origin along the negative real axis with phase $-180°$ as $\omega \to \infty$. This is not consistent with an LQ-optimal loop gain, which will always remain at a distance ≥ 1 from the critical point -1 and will hence have an asympototic phase of $-90°$. Therefore, L cannot be an LQ-optimal state feedback vector.

8.4 The system has two unstable poles in 2 and 3. If the cost function should be less than ∞ then the system must be stabilizable, i.e. all unstable poles must be controllable (due to $Q_1 > 0$). The controllability matrix is given by

$$\mathcal{C} = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} -4 & -12 \\ 28 & 24 \end{pmatrix}$$

which is a rank 1 matrix. Thus, only one of the modes is controllable meaning that there is an uncontrollable, unstable mode, and hence, we can not make the cost function less than ∞.

8.5 3) is the only case with a cost on the velocity x_2. This makes the controller try to avoid rapid variations in x_1, so we get $3) - D)$, the only step response without overshoot. The weight, Q_2, on the control signal determines the speed of the system. A low weight on the control signal gives a faster system since we are allowed to use more control signal. This reveals $1) - C)$, $2) - A)$, $4) - B)$.

8.6 **a.** The cost function is

$$J = \int_0^\infty \left(x^T(t) \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} x(t) + \rho u^2(t) \right) dt, \quad \rho = 0.01,\ 10,\ 1000$$

b. See Figure 8.1 for step responses, and Matlab code below.

```
A = [1 3; 4 8]; B = [1; 0.1]; C = [0 1];
P = ss(A,B,C,0);

Q1 = [1 0; 0 2]; Q2_vector = [0.01 1 100];

clf
for i=1:length(Q2_vector)
[L,S,E] = lqr(P,Q1,Q2_vector(i));

% Calculating Lr (static gain to output should be 1)
Lr = 1/(C/(B*L-A)*B);

% Closed loop from r to u:
Gur = ss(A-B*L,B*Lr,-L,Lr);

% Closed loop from r to y:
```

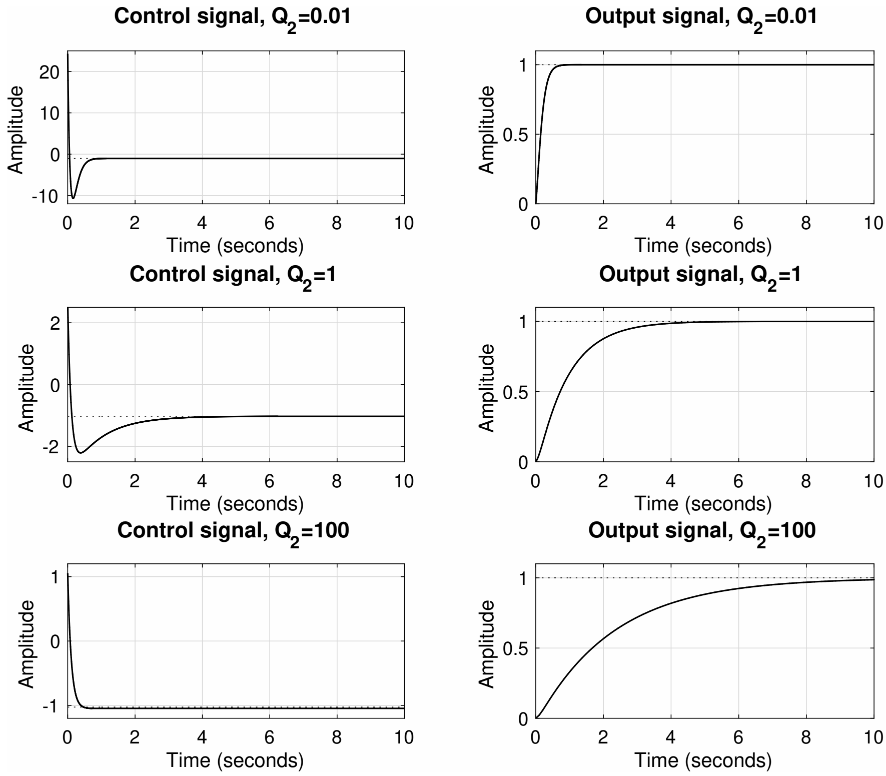

Figure 8.1 Step responses for different weight on control signal.

```
Gyr = ss(A-B*L,B*Lr,C,0);

% Plotting step responses
subplot(3,2,i*2-1)
step(Gur)
axis([0 10 -Inf Inf])
title(['Control signal, Q_2=' num2str(Q2_vector(i))])
subplot(3,2,i*2)
step(Gyr)
axis([0 10 -Inf Inf])
title(['Output signal, Q_2=' num2str(Q2_vector(i))])
poles{i} = E;
end
poles{:}
```

8.7 **a.** Weighting matrices $Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $Q_2 = \eta$. The Riccati equation to be solved with respect to S is

$$A^T S + SA + Q_1 - SBQ_2^{-1}B^T S = 0$$

Put

$$S = \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix},$$

which gives

$$\begin{pmatrix} 0 & 0 \\ s_1 & s_2 \end{pmatrix} + \begin{pmatrix} 0 & s_1 \\ 0 & s_2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \frac{1}{\eta} \cdot \begin{pmatrix} s_2^2 & s_2 s_3 \\ s_2 s_3 & s_3^2 \end{pmatrix} = 0$$

We see, by insertion, that

$$\begin{aligned} s_1 &= \sqrt{2} \cdot \eta^{1/4} \\ s_2 &= \eta^{1/2} \\ s_3 &= \sqrt{2} \cdot \eta^{3/4} \end{aligned}$$

solves the Riccati equation.

b. The optimal state feedback is

$$\begin{aligned} L = Q_2^{-1} B^T S &= \frac{1}{\eta} \cdot \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{2}\eta^{1/4} & \eta^{1/2} \\ \eta^{1/2} & \sqrt{2} \cdot \eta^{-3/4} \end{pmatrix} \\ &= \frac{1}{\eta} \cdot \begin{pmatrix} \eta^{1/2} & \sqrt{2}\eta^{3/4} \end{pmatrix} = \begin{pmatrix} \eta^{-1/2} & \sqrt{2} \cdot \eta^{-1/4} \end{pmatrix} \end{aligned}$$

The poles are the eigenvalues to $A - BL$. Put $\mu = \eta^{-1/4} \Rightarrow L = \begin{pmatrix} \mu^2 & \sqrt{2} \cdot \mu \end{pmatrix}$. This gives

$$0 = \det \begin{pmatrix} s & -1 \\ \mu^2 & s + \sqrt{2} \cdot \mu \end{pmatrix} = s^2 + \sqrt{2}\mu s + \mu^2,$$

that is

$$\begin{aligned} s &= -\frac{\mu}{\sqrt{2}} \pm \sqrt{\frac{\mu^2}{2} - \mu^2} = -\frac{\mu}{\sqrt{2}} \pm i \cdot \frac{\mu}{\sqrt{2}} = \\ &= -\frac{\mu}{\sqrt{2}} \cdot (1 \pm i) = -\frac{1}{\sqrt{2} \cdot \eta^{1/4}} \cdot (1 \pm i) \end{aligned}$$

If η is reduced, the distance between the poles and the origin will increase. This means that the size of $u(t)$ will increase.