

FRTN10 Exercise 9. Kalman Filtering

9.1 Consider the unstable first-order system

$$\begin{aligned}\dot{x}(t) &= x(t) + u(t) + w_1(t) \\ y(t) &= x(t) + w_2(t)\end{aligned}$$


The uncorrelated noise signals $w_i(t)$ are white with intensities R_i . We want to investigate how the optimal Kalman filter depends on noise parameters.

- Show that the Kalman filter gain only depends on the ratio $\beta = R_1/R_2$.
- Find the observer error dynamics, i.e., the dynamics of the estimation error $\tilde{x}(t) = x(t) - \hat{x}(t)$.
- How does the error dynamics depend on the ratio $\beta = R_1/R_2$? Interpret the result for large β (process noise much larger than measurement noise), and for small β (measurement noise much larger than process noise).

9.2 A Kalman filter should be designed for the second-order system

$$\begin{aligned}\dot{x}(t) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t) + w_1(t) \\ y(t) &= (1 \ 0) x(t) + w_2(t)\end{aligned}$$

where w_1 and w_2 are uncorrelated white noise processes with intensities $R_1 = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ and $R_2 = 1$, respectively.

- Calculate the minimum observer error covariance P and the optimal Kalman filter gain K .
- Write down the resulting filter equations for \hat{x}_1 and \hat{x}_2 .
- (*)  Find the minimum error covariance P and the optimal filter gain K using `lqe` in Matlab.

9.3 Consider an integrator process driven by unit intensity white noise:

$$\dot{x}(t) = w_1(t), \quad R_1 = 1$$

For each of the cases below, assuming that an optimal Kalman filter should be designed, compute the minimum observer error variance.

- There is one noisy measurement of x , given by

$$y(t) = x(t) + w_2(t), \quad R_2 = 1$$

- There are two independent noisy measurements of x , given by

$$\begin{aligned}y_1(t) &= x(t) + w_{21}(t), & R_{21} &= 1 \\ y_2(t) &= x(t) + w_{22}(t), & R_{22} &= 10\end{aligned}$$

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- c. There are two dependent noisy measurements of x , given by

$$\begin{aligned} y_1(t) &= x(t) + w_{21}(t), \\ y_2(t) &= x(t) + w_{22}(t), \end{aligned} \quad R_2 = \begin{pmatrix} 1 & 1 \\ 1 & 10 \end{pmatrix}$$

- 9.4 We would like to design an output feedback controller for the stable second-order system

$$G(s) = \frac{1}{(s+1)^2}.$$

The first step is to design a Kalman filter. The process and its disturbances have been modeled as shown in Figure 9.1, where w_{11} , w_{12} and w_2 are uncorrelated, unit intensity white noise processes. A low-frequency input disturbance has been modeled by filtering w_{12} through a low-pass filter $\frac{1}{s+\epsilon}$, where $\epsilon > 0$ is a small number.

Write down the system in state-space form and find all the relevant matrices needed to state a Riccati equation for the Kalman filter.

- 9.5 Consider control of a DC-motor,

$$G(s) = \frac{1}{s(s+1)}$$

Introduce the state variables $x_1 = y$, $x_2 = \dot{y}$. White process noise is active on both states with intensity 1 and with input vector $(0.1 \ 0.1)^T$. There is also noise on the measurements with intensity 0.1. This gives the following state-space model

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t) + \begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix} v_1(t) \\ y(t) &= (1 \ 0) x(t) + v_2(t) \end{aligned}$$

with $R_1 = 1$, $R_2 = 0.1$ and $R_{12} = 0$

- a. The motor will be connected to an external system that might be oscillatory around the frequency 0.5 rad/s, but there is no detailed knowledge about its properties. In order not to excite the oscillatory modes we would like the

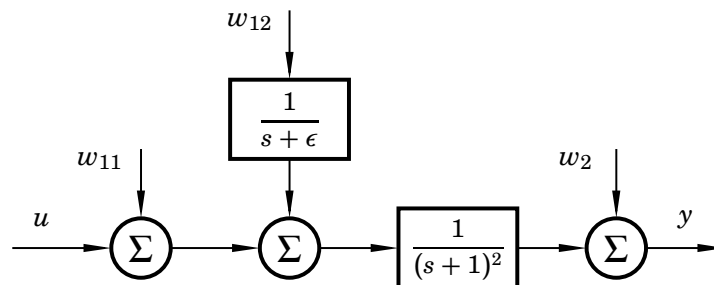


Figure 9.1 Process with additive low-frequency input disturbance.

controller to have small gain around the this frequency. This can be achieved by extending the measurement equation as

$$y_e(t) = (1 \ 0)x(t) + v_2(t) + v_3(t)$$


The extra measurement disturbance v_3 is generated by passing unit intensity white noise n through a second-order filter with a transfer function

$$H(s) = \frac{K_v s}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

with $\omega_0 = 0.5$ rad/s. The parameter ζ determines the magnitude of the filter resonance peak, and we can choose e.g. $\zeta = 0.02$. Derive the extended state-space model

$$\begin{aligned} \dot{x}_e(t) &= A_e x_e(t) + B_e u(t) + N_e \begin{pmatrix} v_1(t) \\ n(t) \end{pmatrix} \\ y_e(t) &= C_e x_e(t) + v_2(t) \end{aligned}$$

and the associated noise intensity matrices needed to compute the Kalman filter.

- b.** (*)  Compute the Kalman filter using `kalman` in Matlab. Plot the transfer function of the Kalman filter from y to $\hat{x}_1 (= \hat{y})$. Can you see the implication of the noise modeling?

9.6 (*) Consider the task of estimating the states of a double integrator where noise with intensity 1 affects the input only and we have measurement noise of intensity 1.

- a.** Determine the optimal Kalman filter.
b. What are the Kalman filter poles?

Solutions to Exercise 9. Kalman Filtering

9.1 a. With the problem parameters $A = C = 1$, the Riccati equation reduces to

$$2P + R_1 - \frac{P^2}{R_2} = 0,$$

which has the positive solution $P = R_2 + R_2\sqrt{1 + \frac{R_1}{R_2}}$. Thus, the Kalman filter gain is

$$K = \frac{1}{R_2}P = 1 + \sqrt{1 + \frac{R_1}{R_2}} = 1 + \sqrt{1 + \beta}.$$

b. The Kalman filter error dynamics are given by

$$\begin{aligned}\dot{\hat{x}}(t) &= (A - KC)\hat{x}(t) + w_1(t) - Kw_2(t) \\ &= -\sqrt{1 + \beta}\hat{x}(t) + w_1(t) - (1 + \sqrt{1 + \beta})w_2(t)\end{aligned}$$

c. The position of the Kalman filter pole is $-\sqrt{1 + \beta}$. We can see that if $\beta \rightarrow \infty$, the pole of the Kalman filter $\rightarrow -\infty$. Hence, the estimation error dynamics are fast, and the Kalman filter very much trusts the measurements. On the other hand, if $\beta \rightarrow 0$, the Kalman filter pole tends to -1 , that is, as fast as the process pole. Now, the filter trusts the process model much more than the measurements.

9.2 a. With the problem parameters

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad C = (1 \ 0), \quad R_1 = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \quad R_2 = 1, \quad P = \begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix}$$

the Riccati equation $AP + PA^T + R_1 - PC^T R_2^{-1} CP = 0$ leads to

$$\begin{aligned}-p_1^2 + 2p_2 + 3 &= 0 \\ p_1 + p_3 - p_1 p_2 &= 0 \\ -p_2^2 + 2p_2 + 3 &= 0\end{aligned}$$

with the positive solution $p_1 = p_2 = 3, p_3 = 6$. The optimal P and K are thus

$$P = \begin{pmatrix} 3 & 3 \\ 3 & 6 \end{pmatrix}, \quad K = PC^T = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

b. The Kalman filter is given by $\frac{d\hat{x}}{dt} = (A - KC)\hat{x} + Bu + Ky$. Inserting the problem data and the optimal K gives

$$\begin{aligned}\frac{d\hat{x}_1}{dt} &= -3\hat{x}_1 + \hat{x}_2 + u + 3y \\ \frac{d\hat{x}_2}{dt} &= -2\hat{x}_1 + 3y\end{aligned}$$

c. See Matlab code below.

```
>> A = [0 1; 1 0];
>> B = [1; 0];
>> C = [1 0];
>> R1 = [3 0; 0 3];
>> R2 = 1;
>> [K,P] = lqe(A,eye(2),C,R1,R2)
K =
    3.0000
    3.0000
P =
    3.0000    3.0000
    3.0000    6.0000
```

9.3 In each case, we are looking for the observer error covariance $E \hat{x}^2 = P$, where P is given by the solution to the algebraic Riccati equation

$$AP + PA^T + R_1 - (PC^T + R_{12})R_2^{-1}(PC^T + R_{12})^T = 0$$

a. In this case we have $A = 0$, $C = 1$, $R_1 = R_2 = 1$, $R_{12} = 0$ and the Riccati equation becomes

$$1 - P^2 = 0$$

with the solution $P = 1$.

b. In this case we have $A = 0$, $C = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $R_1 = 1$, $R_2 = \begin{pmatrix} 1 & 0 \\ 0 & 10 \end{pmatrix}$, $R_{12} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and the Riccati equation becomes

$$1 - \frac{11}{10}P^2 = 0$$

with the solution $P = \sqrt{\frac{10}{11}} \approx 0.95$. Note that, by adding an independent sensor with large measurement noise, we can still reduce the observer error.

c. In this case we have $A = 0$, $C = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $R_1 = 1$, $R_2 = \begin{pmatrix} 1 & 1 \\ 1 & 10 \end{pmatrix}$, $R_{12} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and the Riccati equation becomes

$$1 - P^2 = 0$$

with the solution $P = 1$. In this case, adding a second sensor does not help in reducing the observer error. An interpretation of the matrix R_2 is that the second sensor measures the same signal as the first sensor, plus some additional noise. Hence, the second signal contains no additional information.

9.4 Taking for instance x_1 as the output of $\frac{1}{s+\epsilon}$ and using x_2 and x_3 to realize $\frac{1}{(s+1)^2}$ we obtain

$$\begin{aligned}\dot{x}_1 &= -\epsilon x_1 + w_{12} \\ \dot{x}_2 &= x_1 - x_2 + u + w_{11} \\ \dot{x}_3 &= -x_3 + x_2 \\ y &= x_3 + w_2\end{aligned}$$

The relevant matrices for stating the Kalman filter Riccati equation are

$$A = \begin{pmatrix} -\epsilon & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}, \quad C = (0 \ 0 \ 1), \quad R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad R_2 = 1$$

- 9.5 a.** We can choose for instance the controllable canonical form to realize the filter $H(s)$:

$$\begin{aligned} \dot{x}_H(t) &= \begin{pmatrix} 0 & 1 \\ -0.25 & -0.02 \end{pmatrix} x_H(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} n(t) \\ v_3(t) &= (0 \ K_v) x_H(t) \end{aligned}$$

Introducing the extended state vector $x_e = \begin{pmatrix} x \\ x_H \end{pmatrix}$ we can write the extended system as

$$\begin{aligned} \dot{x}_e(t) &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -0.25 & -0.02 \end{pmatrix} x_e(t) + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} u(t) + \begin{pmatrix} 0.1 & 0 \\ 0.1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1(t) \\ n(t) \end{pmatrix} \\ y_e(t) &= (1 \ 0 \ 0 \ K_v) x_e(t) + v_2(t) \end{aligned}$$

Thus, we have the intensity matrices $R_1 = \text{diag}(1, 1)$, $R_2 = 0.1$.

- b.** See Figure 9.1 for the Bode plot of the Kalman filter transfer function from measurement $y(t)$ to estimated process output $\hat{x}_1(t)$ using $K_v = 1$. We see a large attenuation of frequencies at $\omega = 0.5$ rad/s. Matlab code:

```
% Extended process model
A = [0 1 0 0; 0 -1 0 0; 0 0 0 1; 0 0 -0.25 -0.02];
B = [0; 1; 0; 0];
Kv = 1;
C = [1 0 0 Kv];
N = [0.1 0; 0.1 0; 0 0; 0 1]; % noise input matrix
R1 = eye(2);
R2 = 0.1;

% Design Kalman filter
sysk = ss(A,[B N],C,0);
kest = kalman(sysk,R1,R2);
Gx1hat_y = kest(2,2); % transfer function from y to x1hat
figure(1)
bode(Gx1hat_y)
```

- 9.6 a.** One possible state-space realization is

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} v_1(t) \\ y(t) &= (1 \ 0) x(t) + v_2(t) \end{aligned}$$

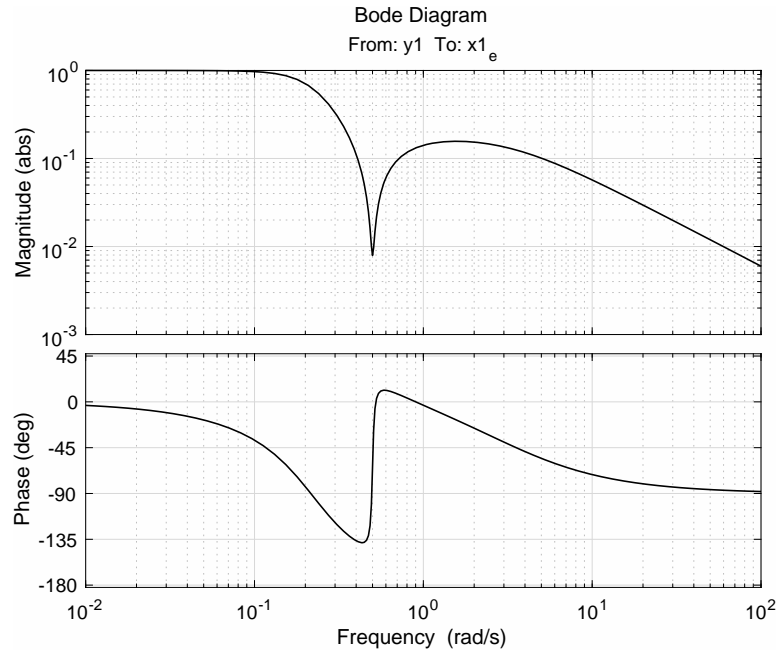


Figure 9.1 Kalman filter Bode diagram in Problem 9.5 b.

The Riccati equation

$$AP + PA^T + R_1 - PC^T R_2^{-1} CP = 0$$

is solved by letting $P = \begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix}$. The equations become

$$\begin{aligned} 2p_2 - p_1^2 &= 0 \\ p_3 - p_1 p_2 &= 0 \\ 1 - p_2^2 &= 0 \end{aligned}$$

The positive solution is

$$P = \begin{pmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{pmatrix}$$

with the optimal gain

$$K = PC^T = (\sqrt{2} \ 1)^T$$

b. The poles of the Kalman filter are the eigenvalues of $A - KC$,

$$A - KC = \begin{pmatrix} -\sqrt{2} & 1 \\ -1 & 0 \end{pmatrix}$$

with the eigenvalues $\lambda_j = \frac{1}{\sqrt{2}}(-1 \pm i)$.