

## FRTN10 Exercise 11. Youla Parametrization, Internal Model Control

**11.1** Consider the control system in Figure 11.1, designed for the stable, linear, SISO system  $P_0$ . We first want to rewrite the system into the general form in Figure 11.2. In this figure,  $w$  are the external inputs to the system (e.g., disturbances and reference signals),  $z$  gathers all signals that we are interested in controlling,  $u$  are the output signals from  $C$ , and  $y$  contains all signals used by the controller (e.g., reference signals and measurements).

a. Let

$$w = \begin{pmatrix} d \\ n \end{pmatrix}, \quad z = \begin{pmatrix} x \\ v \end{pmatrix}$$

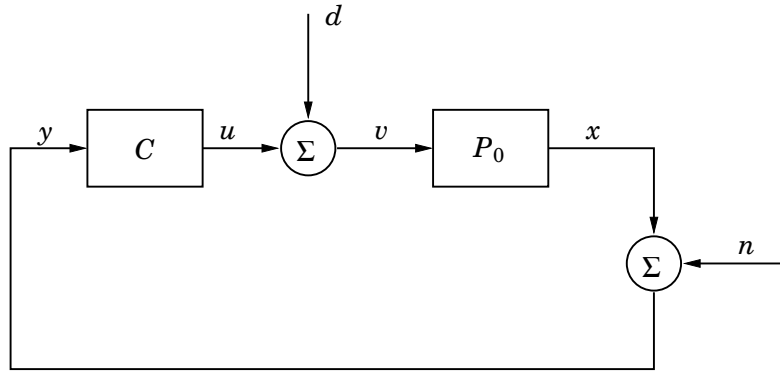
The generalized plant  $P$  in Figure 11.2 consists of four subsystems:

$$P = \begin{pmatrix} P_{zw} & P_{zu} \\ P_{yw} & P_{yu} \end{pmatrix}$$

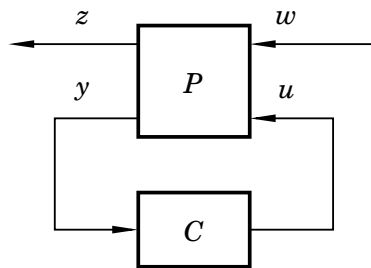
Determine the transfer matrices  $P_{zw}$ ,  $P_{zu}$ ,  $P_{yw}$  and  $P_{yu}$ .

b. The closed-loop system from  $w$  to  $z$  is denoted  $G_{zw}$ . Show that  $G_{zw} = P_{zw} + P_{zu}C(I - P_{yu}C)^{-1}P_{yw}$  for the general system in Figure 11.2.

c. Determine  $G_{zw}$  for the system in Figure 11.1. Introduce the Youla parameter  $Q = \frac{C}{1 - P_{yu}C}$  and show that each element of  $G_{zw}$  is an affine function of  $Q$ .



**Figure 11.1** Original block diagram in Problem 11.1.



**Figure 11.2** General closed-loop system in Problems 11.1 and 11.2.

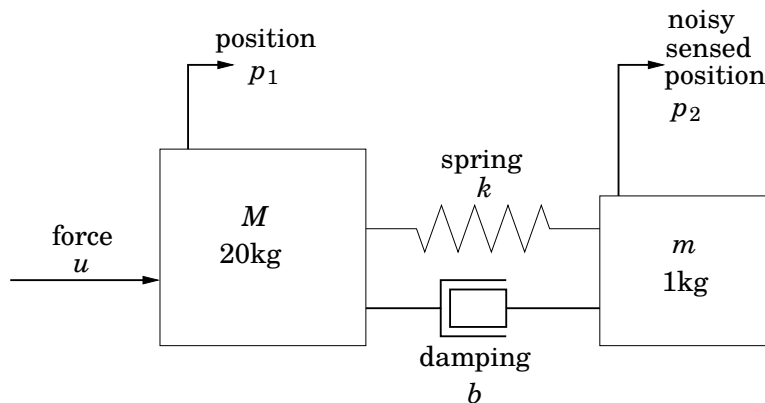


Figure 11.3 Mass-spring system in Problem 11.2.

11.2 Note: You should have solved this problem before you start working on Exercise 12.

Let us consider the physical system shown in Figure 11.3, showing two masses, lightly coupled through a spring with spring constant  $k$  and damping  $b$ . The only sensor signal we have is the noisy measurement  $p_2 + n$  of the position,  $p_2$ , for the small mass,  $m$ . The purpose of the controller is to make the position of the large mass,  $p_1$ , follow a reference input,  $r$ , such that the control error  $e$  becomes small, while not spending too large control effort. This is expressed in a quadratic cost function (the objective) as

$$J = \int_0^{\infty} (\gamma e^2(t) + \rho u^2(t)) dt$$

where  $e$  and  $u$  are generated in response to a reference change  $r$ .

Minimization of this function will be subject to constraints on:

- the magnitude of the control signal  $|u(t)| < u_{max}$  (the force acting on the large mass) during a reference step
- step response overshoot, rise time and settling time from  $r$  to the position  $p_1$  (performance constraint)
- the maximum norm of the sensitivity function,  $\|S(i\omega)\|_{\infty} \leq M_s$  (robustness constraint)

The system can be described by the equations of motion

$$\begin{aligned} M\ddot{p}_1 + b(\dot{p}_1 - \dot{p}_2) + k(p_1 - p_2) &= u \\ m\ddot{p}_2 + b(\dot{p}_2 - \dot{p}_1) + k(p_2 - p_1) &= 0 \end{aligned}$$

Introducing the state vector

$$x = \begin{pmatrix} \dot{p}_1 \\ p_1 \\ \dot{p}_2 \\ p_2 \end{pmatrix}$$

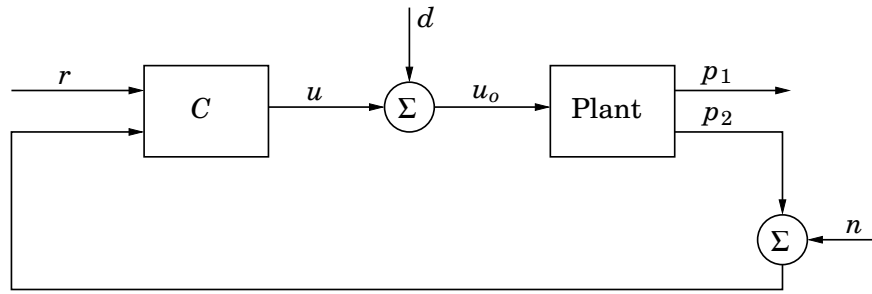


Figure 11.4 The block diagram of the control system in Problem 11.2.

we can write the system in state-space form as

$$\begin{aligned} \dot{x} &= Ax + Bu \\ p_1 &= C_1x \\ p_2 &= C_2x \end{aligned}$$

where

$$A = \begin{pmatrix} -b/M & -k/M & b/M & k/M \\ 1 & 0 & 0 & 0 \\ b/m & k/m & -b/m & -k/m \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 1/M \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}$$

$$C_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}$$

Let  $M = 20$  kg,  $m = 1$  kg,  $k = 32$  N/m and  $b = 0.3$  Ns/m.

Now, consider the problem to set up this system in a form such that we can optimize over the  $Q$  parametrization. Have Figure 11.4 as a reference. Then, the exogenous inputs,  $w$ , of the system are

- the reference  $r$ ,
- the measurement noise  $n$ ,
- the input disturbance  $d$ .

The performance outputs,  $z$ , are

- the position  $p_1$  of the large mass  $M$ ,
- the input to the plant,  $u_o$ ,
- the control error  $e = r - p_1$ .

The control signal,  $u$ , to the plant is

- the force  $u$  on the large mass  $M$ .

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The controller inputs  $y$  are

- the reference  $r$ ,
- the noisy measurement  $p_2 + n$ .

In other words, we have

$$w = \begin{pmatrix} r \\ n \\ d \end{pmatrix}, \quad z = \begin{pmatrix} p_1 \\ u_o \\ e \end{pmatrix}, \quad y = \begin{pmatrix} r \\ p_2 + n \end{pmatrix}.$$

With these variables, we can rewrite the system in the general form shown in Figure 11.2. In state-space form, this becomes

$$\dot{x} = Ax + B_w w + Bu \quad (11.1)$$

$$z = C_z x + D_{zw} w + D_{zu} u \quad (11.2)$$

$$y = C_y x + D_{yw} w + D_{yu} u \quad (11.3)$$

- Determine all matrices in equations (11.1)–(11.3).
- In the next exercise session we will use software that solves the minimization problem. This software will need to know the general process  $P$ , determined by (11.1)–(11.3), and the element indices of the closed-loop transfer function  $G_{zw}$  corresponding to the constraints and cost function specified for the control design problem. For instance, the step response overshoot, rise time and settling time will correspond to  $G_{p_1 r}$  which has index  $(1, 1)$ . Determine the rest of these indices, i.e., for
  - the control signal magnitude constraint,
  - the sensitivity function magnitude constraint,
  - the objective function.
- Introducing the Youla parameter  $Q = C(I - P_{yu}C)^{-1}$ , how many inputs and outputs will  $Q$  have?

**11.3** Derive a controller using the IMC method for the following system:

$$P(s) = \frac{6 - 3s}{s^2 + 5s + 6}.$$

Use the method of mirroring the RHP zero when forming  $Q$  and place any extra closed-loop poles in  $-1/\lambda$ . Show that the controller has the form of a PID controller with a first-order measurement filter, i.e.,

$$K \left( 1 + \frac{1}{T_i s} + T_d s \right) \frac{1}{sT + 1}$$

**11.4** Processes in industry often have time delays that limit the achievable performance. Consider the simple process

$$P(s) = \frac{1}{s + 1} e^{-4s},$$



which is a delay dominant process (i.e., the time delay is longer than the time constant).

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- a.** Use IMC to design a delay-compensating controller, using the approach of ignoring the time delay when  $Q(s)$  is calculated.
- b.** Use IMC to design a delay-compensating controller, using the approach of approximating the time delay with a first order Padé approximation,

$$e^{-sL} \approx \frac{1 - sL/2}{1 + sL/2},$$

and then ignoring the unstable zero when  $Q(s)$  is calculated.

- c. (\*)**   For both cases above, draw the Nyquist plot for the loop transfer function when  $\lambda = 3$  and conclude whether the closed-loop system is stable.

## Solutions to Exercise 11. Youla Parametrization, Internal Model Control

11.1 a. We can divide  $P$  even further into smaller parts such that

$$P_{zw} = \begin{pmatrix} P_{xd} & P_{xn} \\ P_{vd} & P_{vn} \end{pmatrix}, \quad P_{zu} = \begin{pmatrix} P_{xu} \\ P_{vu} \end{pmatrix}, \quad P_{yw} = \begin{pmatrix} P_{yd} & P_{yn} \end{pmatrix}$$

Looking at the block diagram of the closed-loop system and removing the controller  $C$ , we see that

$$\begin{aligned} P_{xd} &= P_0, & P_{xn} &= 0, & P_{vd} &= 1, & P_{vn} &= 0 \\ P_{xu} &= P_0, & P_{vu} &= 1 \\ P_{yd} &= P_0, & P_{yn} &= 1 \\ P_{yu} &= P_0 \end{aligned}$$

This gives us the transfer matrix

$$P = \begin{pmatrix} P_0 & 0 & P_0 \\ 1 & 0 & 1 \\ P_0 & 1 & P_0 \end{pmatrix},$$

so that

$$P_{zw} = \begin{pmatrix} P_0 & 0 \\ 1 & 0 \end{pmatrix}, \quad P_{zu} = \begin{pmatrix} P_0 \\ 1 \end{pmatrix}, \quad P_{yw} = \begin{pmatrix} P_0 & 1 \end{pmatrix}, \quad P_{yu} = P_0$$

**b.**

$$u = Cy$$

$$y = P_{yu}u + P_{yw}w = P_{yu}Cy + P_{yw}w \Rightarrow y = (I - P_{yu}C)^{-1}P_{yw}w$$

$$z = P_{zw}w + P_{zu}u = P_{zw}w + P_{zu}Cy = (P_{zw} + P_{zu}C(I - P_{yu}C)^{-1}P_{yw})w$$

**c.** Using the result from **b.** we get

$$\begin{aligned} G_{zw} &= \begin{pmatrix} P_0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} P_0 \\ 1 \end{pmatrix} C(1 - P_0C)^{-1} \begin{pmatrix} P_0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} P_0 & 0 \\ 1 & 0 \end{pmatrix} + \frac{C}{1 - P_0C} \begin{pmatrix} P_0^2 & P_0 \\ P_0 & 1 \end{pmatrix} \\ &= \frac{1}{1 - P_0C} \begin{pmatrix} (P_0 - P_0^2C) + P_0^2C & P_0C \\ (1 - P_0C) + P_0C & C \end{pmatrix} \end{aligned}$$

Introducing  $Q = \frac{C}{1 - P_{yu}C}$  gives

$$G_{zw} = P_{zw} + P_{zu}QP_{yw} = \begin{pmatrix} P_0 & 0 \\ 1 & 0 \end{pmatrix} + Q \begin{pmatrix} P_0^2 & P_0 \\ P_0 & 1 \end{pmatrix} = \begin{pmatrix} P_0 + P_0^2Q & P_0Q \\ 1 + P_0Q & Q \end{pmatrix}$$

where each element of  $G_{zw}$  is affine in  $Q$ .

**11.2 a.** From the equation for the plant and the diagram of the control loop (removing the controller), we can see that the generalized plant is described by

$$\begin{aligned}
 \dot{x} &= Ax + B(u + d) = Ax + \begin{pmatrix} 0 & 0 & B \end{pmatrix} \begin{pmatrix} r \\ n \\ d \end{pmatrix} + Bu \\
 &= Ax + B_w w + Bu \\
 z &= \begin{pmatrix} p_1 \\ u_o \\ e \end{pmatrix} = \begin{pmatrix} C_1 x \\ u + d \\ r - p_1 \end{pmatrix} = \begin{pmatrix} C_1 x \\ u + d \\ r - C_1 x \end{pmatrix} \\
 &= \begin{pmatrix} C_1 \\ 0 \\ -C_1 \end{pmatrix} x + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} r \\ n \\ d \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} u \\
 &= C_z x + D_{zw} w + D_{zu} u \\
 y &= \begin{pmatrix} r \\ p_2 + n \end{pmatrix} = \begin{pmatrix} r \\ C_2 x + n \end{pmatrix} \\
 &= \begin{pmatrix} 0 \\ C_2 \end{pmatrix} x + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} r \\ n \\ d \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} u \\
 &= C_y x + D_{yw} w + D_{yu} u
 \end{aligned}$$

- b.** The constraint on the control signal magnitude,  $|u(t)| \leq u_{max}$ , will correspond to the closed-loop transfer matrix  $G_{u_o r}$ , with index (2, 1). In the original block diagram, we see that the sensitivity function is given by the transfer function  $G_{u_o d}$ . The  $M_s$  constraint will therefore correspond to the index (2, 3). The objective function will be related to two indices, namely those associated with  $G_{er}$  and  $G_{u_o r}$ , (3, 1) and (2, 1).
- c.** Since  $P_{zu}$  is a  $3 \times 1$  system and  $P_{yw}$  is a  $2 \times 3$  system,  $Q$  must be  $1 \times 2$ . Therefore we have that  $Q = [Q_1 \quad Q_2]$ .

**11.3** The system has a zero in the right half-plane, and we use the approach of replacing the unstable zero by its mirrored counterpart when forming  $Q$ . We also need to add a pole to  $Q(s)$  to make it proper, which we place in  $s = -\lambda^{-1}$ . We get

$$Q(s) = \frac{s^2 + 5s + 6}{(6 + 3s)(\lambda s + 1)} = \frac{s + 3}{3(\lambda s + 1)}$$

The controller becomes

$$C(s) = \frac{s^2 + 5s + 6}{s(3\lambda s + 6(\lambda + 1))}$$

which can be rewritten as

$$C(s) = \frac{5}{6(1 + \lambda)} \left( 1 + \frac{6}{5s} + \frac{s}{5} \right) \frac{1}{\frac{3\lambda}{6(\lambda+1)}s + 1}$$

This corresponds to a PID controller in series with a lowpass filter.

11.4 a. We get

$$Q(s) = \frac{(P(s)e^{4s})^{-1}}{\lambda s + 1}$$

Hence, the controller is given by

$$C(s) = \frac{Q(s)}{1 - Q(s)P(s)} = \frac{s + 1}{\lambda s + 1 - e^{-4s}}$$

b. When we calculate the  $Q(s)$ -transfer function, we exclude  $1 - 2s$ . Thus, we then have

$$Q(s) = \frac{(s + 1)(2s + 1)}{(\lambda s + 1)^2}$$

Hence we have the controller

$$C(s) = \frac{Q(s)}{1 - Q(s)P(s)} = \frac{(s + 1)(2s + 1)}{(\lambda s + 1)^2 - (1 + 2s)e^{-4s}}$$

c. The Nyquist plots can be generated in Matlab, using the following lines of code. (Note that feedback delay systems are not always handled by Control System Toolbox.)

```
>> lambda = 3;
>> w = logspace(-2,2,1000);
>> P = 1./(1+i*w).*exp(-4*i*w);
>> Fy1 = (i*w+1)./(lambda*i*w+1-exp(-4*i*w));
>> Fy2 = (i*w+1).*(1+2i*w)./((lambda*i*w+1).*(lambda*i*w+1)-(1+2i*w).*exp(-4*i*w));
>> figure
>> plot(P.*Fy1)
>> grid
>> figure
>> plot(P.*Fy2)
>> grid
```

From the plots (Figure 11.1) we see that neither encircles  $-1$  and the closed-loop system is stable in both cases.

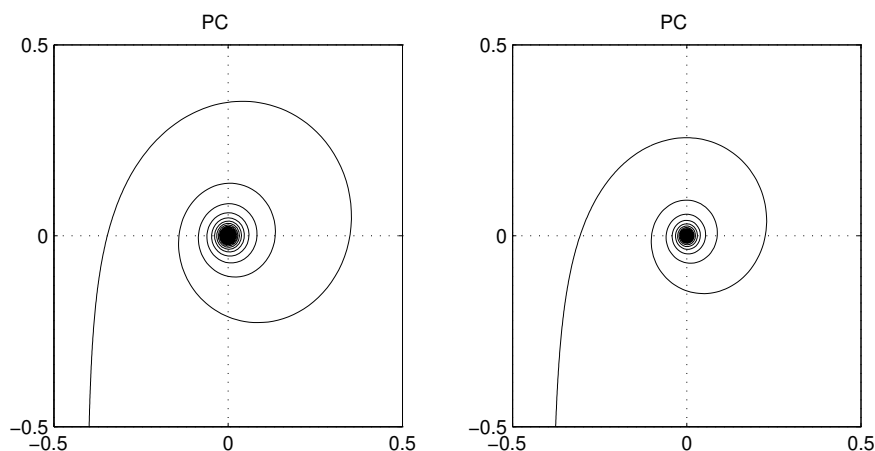


Figure 11.1 Nyquist plots of the loop transfer functions in Problem 11.4 a (left) and b (right).