LUND UNIVERSITY

Department of
AUTOMATIC CONTROL

FRTN15 Predictive Control

Exam 2016-05-11, 14.00–19.00

General Instructions

This is an open book exam. You may use any book you want, including the slides from the lecture, but no exercises, exams, or solution manuals are allowed. Solutions and answers to the problems should be well motivated. The exam consists of 6 problems. The credit for each problem is indicated in the problem. The total number of credits is 25 points. Preliminary grade limits:

- Grade 3: 12-16 points
- Grade 4: 17-21 points
- Grade 5: 22-25 points

Results

The results of the exam will be presented in LADOK, at the latest May 25.

1.

a. Give an intuitive explanation for what a Lyapunov function is. Why can it be used to determine the stability of an arbitrary system? (1 p)

b. Show that a damped pendulum with length l, mass m and damping coefficient k,

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2,$$

is stable at the origin using the Lyapunov canditate

$$V(x) = \frac{g}{l}(1 - \cos(x_1)) + \frac{1}{2}x_2^2.$$

Also, give a simple explanation for what the Lyapunov function describes. (2 p)

Solution

a. A Lyapunov function $V(x)$ describes a quantity in the system that is decreasing with time. We can think of it as an energy function, where the energy is a quantity present in the system that, if it decreases with time, ensures the stability of the system. If $V(x,u)$ contains a controlled variable (control input) u, we can construct a control law that modifies $V(x,u)$ such that it fulfills the requirements of a Lyapunov function. The formal requirements on a Lyapunov function are

- It increases radially with the state vector, i.e. when the state vector is large, the Lyapunov function is large.
- The time derivative $\dot{V}$ is non-positive, i.e. the function does not grow over time.
- It is zero at the origin.

b. The Lyapunov-function time derivative is given by

$$\dot{V} = \frac{g}{l}\dot{x}_1\sin(x_1) + x_2\dot{x}_2 = \frac{g}{l}x_2\sin(x_1) + x_2(-\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2^2) = -\frac{k}{m}x_2^2$$

which is decreasing for all points except along the x_1 axis, however, for these points where $x_1 \neq 0$ (the pendulum end positions), the pendulum will gain velocity which again decreases V until $V = 0$ at the origin. The chosen Lyapunov function V is a weighted sum of the potential and the kinetic energy of the system.

2. Consider the system

$$y(k) = ay(k-1) + u(k-1)$$

where u is an unknown zero-mean white-noise Gaussian process with variance σ_u^2 and a is a constant unknown parameter.

a. Find the recursive least-squares algorithm for estimation of the constant parameter a. Compute the steady-state variance of the parameter-estimate error. (2 p)

b. Now assume that a is time-varying according to

$$a(k+1) = a(k) + v(k)$$

where $v(k)$ is a zero-mean, white-noise Gaussian process with variance σ_v^2. Find the Kalman-filter update equations for the estimation of a. Compute the steady-state variance of the parameter-estimate error. (2 p)

c. State two differences between the RLS and the Kalman-filter approach. (1 p)

Solution

a. The RLS update equations are given by

$$\begin{aligned}
\hat{a}(k) &= \hat{a}(k-1) + P(k-1)\phi(k-1)e(k-1) \\
e(k) &= y(k) - \phi^T(k)\hat{a}(k-1) \\
P(k) &= P(k-1) - \frac{P(k-1)\phi(k)\phi(k)^T P(k-1)}{1 + \phi^T P(k-1)\phi(k)}
\end{aligned}$$

where $\phi(k) = y(k-1)$.

After N samples, the parameter-estimate error $\hat{a} - a$ is given by the expression

$$\hat{a}(k) - a = (\Phi_N^T \Phi_N)^{-1} \Phi_N^T u(k)$$

where the regressor matrix Φ_N is given by

$$\Phi_N^T = \begin{pmatrix} y(1) & \cdots & y(N) \end{pmatrix}^T .$$

The parameter-estimate error variance is given by

$$E\{(\hat{a} - a(k))^2\} = E\{((\Phi_N^T \Phi_N)^{-1} u(k))^2\} = E\{\frac{u(k)^2}{\sum_{k=1}^N y(k)^2}\},$$

and with a first order approximation

$$E\{\frac{u(k)^2}{\sum_{k=1}^N y(k)^2}\} \approx \frac{E\{u(k)^2\}}{E\{\sum_{k=1}^N y(k)^2\}} = \frac{\sigma_u^2}{NE\{y(k)^2\}},$$

furthermore, the variance of y, $E\{y(k)^2\}$ is computed to

$$E\{y(k)^2\} = \frac{\sigma_u^2}{1 - a^2}$$

from the expression

$$y(k) = ay(k-1) + u(k-1),$$

which finally gives the error variance

$$E\{(\hat{a} - a)^2\} = \frac{1 - a^2}{N}.$$

b. For the Kalman filter, the update equations are given by

$$\begin{aligned}
\hat{a}(k) &= \hat{a}(k-1) + K(k)e(k) \\
K(k) &= \frac{P(k-1)\phi(k)}{\sigma_u^2 + \phi(k)^T P(k-1)\phi(k)} \\
e(k) &= y(k) - \phi^T(k)\hat{a}(k-1) \\
P(k) &= P(k-1) - \frac{P(k-1)\phi(k)\phi(k)^T P(k-1)}{\sigma_u^2 + \phi^T P(k-1)\phi(k)} + \sigma_v^2
\end{aligned}$$

and once again, $\phi(k) = y(k-1)$.

In steady state, the stationary-error variance is found by assuming $P_k = P_{k-1}$ in the final Kalman-filter update equation. This gives that

$$P_\infty = \sigma_v^2/2 + \sqrt{(\sigma_v^2/2)^2 + \sigma_v^2(1-a^2)}$$

when inserting the expectation of the regressor squared

$$E\{\phi(k)^2\} = E\{y(k-1)^2\} = \frac{\sigma_u^2}{1-a^2}.$$

c. Two differences between the RLS and the Kalman-filter approach besides that a is assumed to be time-varying in the Kalman filter are that the dynamics of P changes from an exponential to a linear growth rate in the case where $\phi_k = 0$ and that the parameter-error variance does not approach zero as $k \to \infty$.

3. A robot joint with dynamic model

$$G_c(q) = \frac{k(q+1)}{(q-1)^2}$$

should be controlled to follow a repetitive reference signal $y_d(t)$.

a. Find a stable heuristic ILC design $L(q)$ with input-update equation

$$u_{k+1}(t) = u_k(t) + L(q)e_k(t).$$

(2 p)

b. In order to improve robustness it is common to introduce a low-pass filter $Q(q)$ so that

$$u_{k+1}(t) = Q(q)(u_k(t) + L(q)e_k(t))$$

what is the theoretical drawback with this approach? (1 p)

Solution

a. The system is stable for

$$L(q) = k_L \frac{(q-1)^2}{(q+1)}$$

if

$$|1 - G_c(e^{i\omega h})L(e^{i\omega h})| < 1, \; \omega \; \in \; [-\pi, \; \pi], \; \Rightarrow$$
$$|1 - kk_L| < 1$$

which holds if

$$k_L k > 0.$$

and

$$\frac{2}{k} > k_L$$

b. For $Q(q) \neq 1$ the error does not approach zero asymptotically, see page 218 in the lecture notes.

4.

a. Briefly explain the principle of Model Predictive Control. Use a sketch or a diagram to illustrate the terms prediction horizon and control horizon. (1 p)

b. Present a simple "rule-of-thumb" for how many samples the prediction and control horizons should be chosen, why can't they always be chosen arbitrarily long? (1 p)

c. A model predictive controller is designed for control of the double integrator

$$G(s) = 1/s^2.$$

In Fig. 1 we see four different closed-loop set-point responses. Match the responses, A-D, in Fig. 1 with the four different MPC configurations I-IV below, where Q is the tracking-error cost, R is the input-rate cost, u is the system input and H_p and H_c are the prediction and control horizons in number of samples ($h = 0.1s$). Motivate you answers. (3 p)

I	**II**	**III**	**IV**
$Q = 1$	$Q = 1$	$Q = 1$	$Q = 1$
$R = 0.1$	$R = 1$	$R = 0.1$	$R = 0.1$
$\|u\| \leq 1$	$\|u\| \leq 1$	$\|u\| \leq 1.5$	$\|u\| \leq 1$
$H_p = 1$	$H_p = 10$	$H_p = 10$	$H_p = 10$
$H_c = 1$	$H_c = 3$	$H_c = 3$	$H_c = 3$

Solution

a. The receding horizon principle is illustrated in Fig. 2. Given the current state measurement (or estimate), a sequence of future control inputs are determined by minimizing a cost function penalizing predicted inputs and outputs of the system. The first input is implemented, and the optimization is repeated at the next sample.

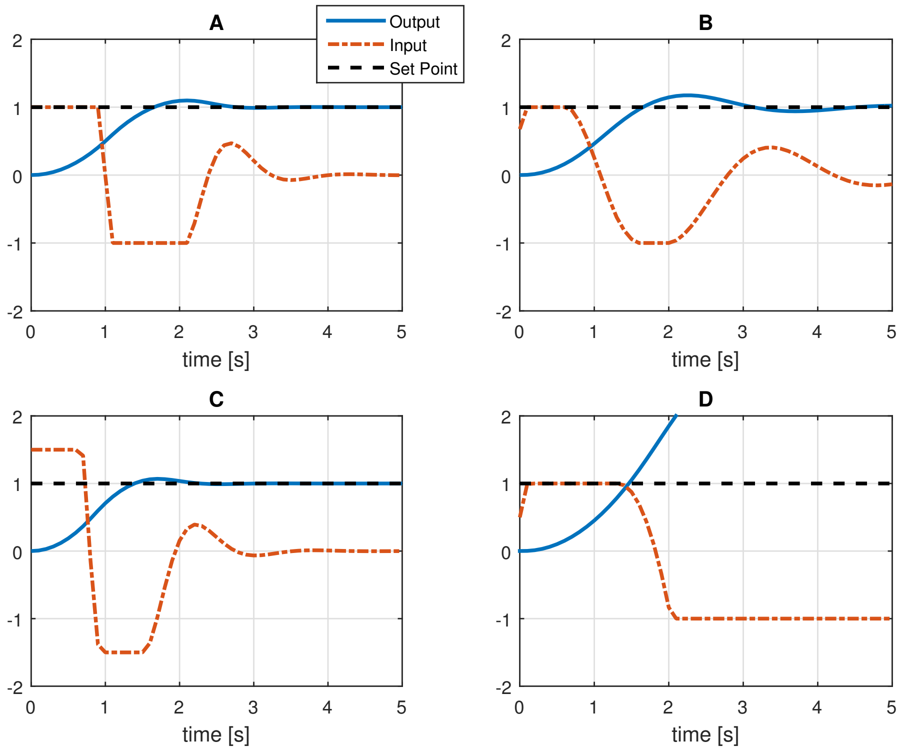

Figure 1 MPC set-point step responses.

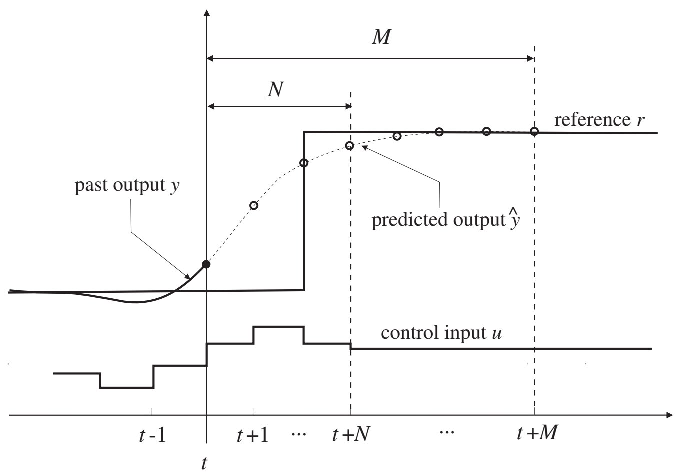

Figure 2 Illustration of the receding horizon principle used in Model Predictive Control.

b. One would like the MPC to overlook the slowest closed-loop dynamics of the system, so if the desired rise-time of the closed-loop is T_c, and the sampling interval h is set according to a common rule of thumb $h = T_c/10$, the prediction horizon should be at least 10 samples.

In MPC, stability can usually be ensured by making the prediction horizon long enough, or even infinite, assuming that the model is correct, the cost matrices are well tuned and input constraints are not too restrictive. However, when making the prediction horizon longer for a given sampling period, the number of unknowns in optimization problem to be solved each sample also grows, which means that computational times might become an issue so that the controller does not have time to solve the optimization problem between samples.

c. We can directly see that response C is the only case where $|u| \geq 1$ which makes it belong to setting III, then, we have two cases with similar settings apart from different input-variation penalty R, increasing R from 0.1 to 1 should give smoother control action with slower step response, this gives that A matches IV and B matches II. Finally, the seemingly unstable behaviour in D comes from setting I with too short prediction and control horizon.

5.

a. Consider the discrete-time model

$$G_0(z) = \frac{b_1 z + b_2}{z^2 + a_1 z + a_2}.$$

Design a minimum-degree model-reference controller (MRC) with zero cancellation, and the reference model

$$G_m(z) = \frac{b_1^m z + b_2^m}{z^2 + a_1^m z + a_2^m}.$$

(2 p)

b. Now, lets assume that $b_1 = 1$ and $b_2 = -2$, why will this cause problems with the approach in a)? How can this problem be solved? (1 p)

c. Describe how the MRC framework could be augmented for the case when the model parameters of G_0 are slowly time varying. (1 p)

Solution

a. This solution follows the procedure found in chapter 9 in the lecture notes, pages 147-152.

We seek a minimum order ARMAX controller on the form

$$R(z)u_k = -S(z)y_k + T(z)u_k^c,$$

we start by introducing the polynomials

$$\begin{aligned} A(z) &= z^2 + a_1 z + a_2 \\ B(z) &= b_0 z + b_1 = B^- B^+, \end{aligned}$$

where we choose $B^- = b_1$ and $B^+ = z + b_2/b_1$. Furthermore

$$\begin{aligned} A^m(z) &= z^2 + a_1^m z + a_2^m \\ B^m(z) &= b_0^m z + b_1^m = B^- B_1^m, \;\Rightarrow\; B_1^m = \frac{b_0^m z + b_1^m}{b_0}. \end{aligned}$$

Compatibility conditions give that $\deg(A^o) = \deg(A) - \deg(B) - 1 = 0$, which gives that $A_o = 1$. The Diophantine equation to solve is thus given by

$$(z^2 + a_1 z + a_2) \times 1 + b_1(s_0 + s_1) = z^2 + a_1^m z + a_2^m$$

where we see that $R_1 = 1$. This yields the following solution for S

$$s_0 = \frac{a_1^m - a_1}{b_1}$$
$$s_1 = \frac{a_2^m - a_2}{b_1}$$

T is then chosen as $T = A_o B_m^1 = (b_0^m z + b_1^m)/(b_0)$, and finally, R is given by $R = R_1 B^+ = B^+ = z + b_2/b_1$.

b. With $b_1 = 1$ and $b_2 = -2$ the system G_0 is non-minimum phase which means that zero cancellation gives an unstable controller, a remedy to this problem is to adjust so that $B^- = z + b_2/b_1$ and $B^+ = b_1$ in order to avoid zero cancellation.

c. The time-varying parameters of G_0 could be estimated using the RLS algorithm with a forgetting factor, which makes the RLS pay less attention to old data. The estimated parameters can then be used to compute the controller parameters, the controller would now be of MRAC type.

6.

a. With the system

$$y(k+1) = ay(k) + \omega(k+1) + c\omega(k),$$

the one-step-ahead predictor

$$\hat{y}(k+1|k) = ay(k)$$

is suggested, compute the prediction error variance. Describe briefly why minimum variance is not obtained by the predictor. (2 p)

b. Design a one-step-ahead minimum variance predictor and compute the corresponding prediction-error variance. (1 p)

c. For the system

$$y(k+1) = ay(k) + bu(k) + \omega(k+1) + c\omega(k).$$

a controller was found by means of pole placement at the origin

$$u(k) = -\frac{a}{b}y(k)$$

yielding the closed-loop output variance

$$E\{y(k)^2\} = (1 + c^2)\sigma_\omega^2,$$

find a causal controller that achieves a lower output variance. (1 p)

Solution

a. With the suggested predictor,

$$E\{(y(k+1) - \hat{y}(k+1|k))^2\} = E\{(\omega(k+1) + c\omega(k))^2\} = (1+c^2)\sigma_\omega^2.$$

The next step output $y(k+1)$ is also depending on old noise input $\omega(k)$ which correlates with the known measurement $y(k)$, this is not taken into account for when using the suggested predictor.

b. The system could be written as

$$y(k+1) = F^*(z^{-1})\omega(k+1) + \frac{G^*(z^{-1})}{C^*(z^{-1})}y(k)$$

where G and F are computed from the Diophantine equation

$$(1 + cz^{-1}) = (1 - az^{-1})f_0 + z^{-1}g_0.$$

This gives $f_0 = 1$, $g_0 = a + c$ and the minimum-variance predictor

$$\hat{y}(k+1) = -b\hat{y}(k) + (a+c)y(k)$$

that achieves the error variance

$$E\{(f_0\omega(k+1))^2\} = \sigma_\omega^2.$$

c. A minimum-variance controller can be found by solving the Diophantine equation

$$C^*(z^{-1}) = A^*(z^{-1})F^*(z^{-1}) + z^{-1}G^*(z^{-1})$$

just as in the previous problem, the minimum-variance controller is then given by

$$u(k) = -\frac{G^*(z^{-1})}{B^*(z^{-1})F^*(z^{-1})}y(k) = -\frac{a+c}{b}y(k)$$

which gives the closed-loop output variance

$$E\{y(k)^2\} = \sigma_\omega^2.$$