LUNDS TEKNISKA HÖGSKOLA
Lunds universitet

Institutionen för
REGLERTEKNIK

FRTN15 Predictive Control

Final Exam October 25, 2013, 08 - 13

General Instructions

This is an open book exam. You may use any book you want, but no notes, exercises, exams, or solution manuals are allowed. Solutions and answers to the problems should be well motivated. The exam consists of 7 problems. The credit for each problem is indicated in the problem. The total number of credits is 25 points. Preliminary grade limits:

Grade 3:	12 – 16 points
Grade 4:	17 – 21 points
Grade 5:	22 – 25 points

Results

The results of the exam will be posted at the latest November 8 on the notice board on the first floor of the M-building.

1. You have been tasked with the estimation of a certain system described by

$$y(t) + ay(t-1) = b_1 u(t-1)y(t-2) + b_2 u(t-2)^2 + d(t)$$

where $d(t)$ is a disturbance that is measurable, and a, b_1 and b_2 are unknown parameters.

a. Derive a linear-in-parameters regression model suitable for estimating the unknown parameters. (2 p)

b. For various reasons you have decided to do online estimation. Furthermore, you know that the parameters are slowly time-varying. Describe your preferred method of estimation for this problem and discuss your methods advantages and drawbacks. (2 p)

Solution

a. By rearranging the terms, collecting measurements of output and disturbance on the left hand side we get

$$\tilde{y}(t) = y(t) - d(t) = -ay(t-1) + b_1 u(t-1)y(t-2) + b_2 u(t-2)^2 = \phi^T(t)\theta$$

with

$$\begin{aligned} \phi(t) &= \begin{pmatrix} -y(t-1) & u(t-1)y(t-2) & u(t-2)^2 \end{pmatrix}^T \\ \theta &= \begin{pmatrix} a & b_1 & b_2 \end{pmatrix}^T \end{aligned}$$

where θ is the parameter vector and $\phi(t)$ the regressors.

b. In order to estimate the parameters online we can use for example the Recursive Least Squares (RLS) algorithm, which updates the parameter estimates in a recursive manner as soon as new measurements are available. The main advantage over the ordinary Least Squares is the fact that we don't need to solve the full set of normal equations for each new measurement. A possible drawback of the RLS is its need of initial guesses on parameter estimates and covariance, which usually are not known a priori. In case the parameters are time-varying, the RLS algorithm can perform poorly since it puts equal weight on all data points. Instead the use of the Kalman filter is recommended, cf. Johansson (4.36)-(4.39). Another possibility is to use RLS combined with a *forgetting factor*, $0 < \lambda \leq 1$, which governs the importance of new data over old data. $\lambda = 1$ produces the ordinary RLS solution, while $\lambda < 1$ gives a solution where new data is deemed more important than old. A smaller λ will give an estimator that is better at following quick parameter changes, at the cost of introducing higher variability in the estimates. There is also the possibility of windup of the estimated covariance matrix P due to too small values of λ that needs to be taken into account.

2. Consider the following system:

$$\begin{aligned} x(k+1) &= 0.2x(k) + u(k) + v(k) \qquad (1) \\ y(k) &= x(k) + e(k) \\ v(k) &\in N(0,1) \\ e(k) &\in N(0,5) \\ x(0) &\in N(0, P_0) \end{aligned}$$

where $N(0, P)$ is a Gaussian variable with mean 0 and variance P.

a. Determine a deadbeat observer for the above system in (1) (i.e., determine the observer gain K such that the observer poles are placed in the origin), for an observer of the following form:

$$\hat{x}(k+1) = A\hat{x}(k) + Bu(k) + K(y(k) - C\hat{x}(k))$$

(1 p)

b. Describe the optimal time-varying Kalman Filter for the system in equation (1). (1 p)

c. Determine the Kalman gain K for the stationary case. (1 p)

d. Which of the two described observers will have the smallest estimation variance $E(\tilde{x}^2)$? (1 p)

Solution

a. The closed-loop polynomial is:

$$\det(z - A + KC) = z - 0.2 + K$$

In order to place all poles in the origin, this polynomial should be equal to z. Hence, the observer gain is $K = 0.2$.

b. The optimal time-varying Kalman filter is

$$\begin{aligned}
\hat{x}(k+1) &= 0.2\hat{x}(k) + u(k) + K_k(y(k)\hat{y}(k)) \\
\hat{y}(k) &= \hat{x}(k) \\
K_k &= \frac{0.2P_{k|k-1}}{5 + P_{k|k-1}} \\
P_{k+1|k} &= 0.2^2 P_{k|k-1} + 1 - \frac{0.2^2 P_{k|k-1}^2}{5 + P_{k|k-1}}
\end{aligned}$$

c. Stationarity means that $P_{k|k-1} = P_{k+1|k} = P$. With this, we have

$$\begin{aligned}
P &= 0.2^2 P + 1 - \frac{0.2^2 P^2}{5 + P} \\
&\Leftrightarrow P^2 + 3.8P - 5 = 0
\end{aligned}$$

The positive solution of this quadratic equation is $P = 1.0343$. With $K = (0.2P)/(5 + P)$, we have $K = 0.0343$.

d. For the deadbeat observer we have:

$$\begin{aligned}
\tilde{x}^2 &= 0.2x(k) + u(k) + v(k) - 0.2\hat{x}(k) - 0.2(x(k) + e(k) - \hat{x}(k)) \\
&= 0.2^2 e(k) + v(k),
\end{aligned}$$

so that $E(\tilde{x}^2) = 0.2^2 \cdot 5 + 1 = 1.2$.
For the Kalman Filter we have $E(\tilde{x}^2) = P = 1.0343$, which is smaller than for the deadbeat observer.

3. The following system is to be controlled using Model Predictive Control

$$\begin{aligned} x(k+1) &= 0.5x(k) + u(k) \\ y(k) &= x(k) \end{aligned} \tag{2}$$

a. Explain the principle of *receding horizon* as used in Model Predictive Control. Make a sketch where you explain what *prediction horizon* and *control horizon* are. (1 p)

b. The prediction of the future output trajectory of the system in (2) can be expressed as:

$$\mathcal{Y}_{H_p} = \mathcal{S}_x x(k) + \mathcal{S}_{u-1} u(k-1) + \mathcal{S}_{\Delta u} \Delta \mathcal{U}_{H_p}.$$

Calculate the matrices $\mathcal{S}_x$, $\mathcal{S}_{u-1}$ and $\mathcal{S}_{\Delta u}$ for $H_p = H_u = 3$. What are the interpreations of these matrices? (2 p)

c. The output of the system (2) should be constrained according to:

$$-1 \leq y(k) \leq 5.$$

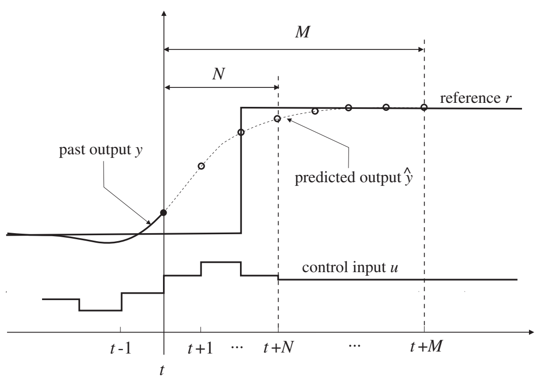
Show that this can be expressed as a constraint on $\Delta u(k)$, where $\Delta u = u(k) - u(k-1)$, according to:

$$-3.5 \leq \Delta u(k) \leq 2.5.$$

You can assume that the current state is $x(k) = 3$ and the previous control signal was $u(k-1) = 1$. (1 p)

Solution

a.



Figur 1 Illustration of the MPC principle.

Receding horizon: The receding horizon principal involves finding an open-loop control sequence that minimizes a certain cost function over a finite horizon. This procedure is performed each time new measurements are available. The first step in the computed control sequence is used as the control signal. Although the computed control sequences are open-loop sequences, the calculation of a new sequence at each sample can be thought of as providing feedback. (gives 0.5 Points)

Prediction horizon and *control horizon*: See Figure 1. The finite horizon over which the cost function is evaluated is the prediction horizon. The cost function depends on the predicted values of the output given the initial state and future control changes, and thus the predictions have to be determined for the prediction horizon. The number of control changes determined each time a new measurement is available is the control horizon. The control horizon can be smaller than the prediction horizon. (gives 0.5 points)

b. Iterating the state equation gives, with $\Delta u(k) = u(k) - u(k-1)$:

$$
\begin{aligned}
x(k+1) &= 0.5x(k) + u(k) \\
&= 0.5x(k) + u(k-1)\Delta u(k) \\
x(k+2) &= 0.5^2 x(k) + 1.5u(k-1) + 1.5\Delta u(k) + \Delta u(k+1) \\
x(k+3) &= 0.5^3 x(k) + 1.75u(k-1) + 1.75\Delta u(k) + 1.5\Delta u(k+1) + \Delta u(k+2)
\end{aligned}
$$

With $y(k) = x(k)$ we have:

$$
\begin{bmatrix} y(k+1) \\ y(k+2) \\ y(k+3) \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5^2 \\ 0.5^3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1.5 \\ 1.75 \end{bmatrix} u(k-1) + \begin{bmatrix} 1 & 0 & 0 \\ 1.5 & 1 & 0 \\ 1.75 & 1.5 & 1 \end{bmatrix} \begin{bmatrix} \Delta u(k) \\ \Delta u(k+1) \\ \Delta u(k+2) \end{bmatrix},
$$

so that

$$
\mathcal{Y}_{H_p} = \mathcal{S}_x x(k) + \mathcal{S}_{u-1} u(k-1) + \mathcal{S}_{\Delta u} \Delta \mathcal{U}_{H_p}.
$$

with

$$
\mathcal{S}_x = \begin{bmatrix} 0.5 \\ 0.5^2 \\ 0.5^3 \end{bmatrix}, \quad \mathcal{S}_{u-1} = \begin{bmatrix} 1 \\ 1.5 \\ 1.75 \end{bmatrix}
$$

$$
\mathcal{S}_{\Delta u} = \begin{bmatrix} 1 & 0 & 0 \\ 1.5 & 1 & 0 \\ 1.75 & 1.5 & 1 \end{bmatrix}
$$

(gives 1 point)

$\mathcal{S}_x$: It has the structure of an observability matrix and represents the component of the predicted future outputs which are obtained from the current state.

$\mathcal{S}_{u-1}$: It represents the propagation of the previous samples control signal $u(k1)$ through the prediction horizon. If the control changes over the horizon are zero, then the predicted outputs are given by $\mathcal{Y}_{H_p} = \mathcal{S}_x x(k) + \mathcal{S}_{u-1} u(k-1)$.

$\mathcal{S}_{\Delta u}$: This matrix is a lower triangular Toeplitz matrix which describes the effects of the sequence of control changes on the predicted outputs.

(gives 1 point)

c. Using equation (2), the output of the system at time $k+1$ can be expressed as:

$$\begin{aligned} y(k+1) &= 0.5x(k) + u(k-1) + \Delta u(k) \\ &= 2.5 + \Delta u(k) \end{aligned}$$

A constraint on $y(k)$ at time k is also valid at time $k+1$. Therefore:

$$\begin{aligned} -1 &\leq y(k) \leq 5 \\ -1 &\leq y(k+1) \leq 5 \\ -1 &\leq 2.5 + \Delta u(k) \leq 5 \\ -3.5 &\leq \Delta u(k) \leq 2.5 \end{aligned}$$

4. In this problem we consider the system

$$y_{k+1} - 1.2y_k + 0.7y_{k-1} = u_k - 0.7u_{k-1} + e_{k+1} + 0.5e_k$$

where e_k is a Gaussian white noise process with variance σ^2.

a. Derive the controller that minimizes the cost function $\mathcal{E}\{(y_{k+1|k})^2\}$. Also, state clearly the value attained for the cost function. (2 p)

b. Derive the controller that minimizes the cost function $\mathcal{E}\{(y_{k+2|k})^2\}$. Also, state clearly the value attained for the cost function. (2 p)

c. Assume now that the zero polynomial of the system is changed to $z-2$. Give an explanation why the previously designed controllers will not work. (1 p)

Solution

This is a minimum variance control problem. First we identify that the system can be written on the form

$$y_{k+1} = \frac{B^*(z^{-1})}{A^*(z^{-1})}u_k + \frac{C^*(z^{-1})}{A^*(z^{-1})}e_{k+1} \tag{3}$$

with

$$\begin{aligned} A^*(z^{-1}) &= 1 - 1.2z^{-1} + 0.7z^{-2} \\ B^*(z^{-1}) &= 1 - 0.7z^{-1} \\ C^*(z^{-1}) &= 1 + 0.5z^{-1} \end{aligned}$$

Solving the Diophantine equation

$$C = AF + z^{-1}G \tag{4}$$

with $\deg F = d - 1 = 0$ (where d is the input-output delay) and $\deg G = n - 1 = 1$ (where n is the system order) yields

$$\begin{aligned} F(z^{-1}) &= 1 \\ G(z^{-1}) &= 1.7 - 0.7z^{-1} \end{aligned}$$

Substituting (4) into (3) gives after some calculations

$$y_{k+1} = Fe_{k+1} + \frac{G}{C}y_k + \frac{BF}{C}u_k$$

where we then can choose the minimum variance controller as $u_k = -G/(BF)y_k$ to get $y_{k+1} = Fe_{k+1}$. The cost function, which is the variance, is given by

$$\mathcal{E}\{(y_{k+1})^2\} = \mathcal{E}\{(Fe_{k+1})^2\} = \sigma^2$$

a. For the two-step ahead minimum variance controller we solve the Diophantine equation

$$C = AF + z^{-2}G$$

with orders $\deg F = 1$ and $\deg G = 1$. Calculations give

$$F(z^{-1}) = 1 + 1.7z^{-1}$$
$$G(z^{-1}) = 1.34 - 1.19z^{-1}$$

The minimum variance controller is again given by $u_k = -G/(BF)y_k$ and the variance of the output becomes

$$\mathcal{E}\{(y_{k+1})^2\} = \mathcal{E}\{(Fe_{k+1})^2\} = 3.89\sigma^2$$

b. The system is now non-minimum phase (the zero is located outside the unit circle). A minimum variance controller cancels the process zeroes, which implies that they end up as poles in the controller. An unstable zero, as in this case, would end up as an unstable pole in the controller, thus making the system unstable.

5. In this problem we consider adaptive control of the system

$$H(z) = \frac{b_0 z + b_1}{z^2 + a_1 z + a_2}$$

with the reference model

$$H_m(z) = \frac{b_0 z + b_1}{z^2 + a_{m1} z + a_{m2}}$$

Design a minimum degree indirect adaptive controller. You may use the observer polynomial $A_o(z) = z^n$ for some n. Outline how you estimate the parameters of the system (2 p)

Solution

We seek a minimum order ARMAX controller on the form

$$R(z)u_k = -S(z)y_k + T(z)u_k^c$$

We start by introducing the polynomials

$$\begin{aligned} A(z) &= z^2 + a_1 z + a_2 \\ B(z) &= b_0 z + b_1 = B^- B^+ \\ A_m(z) &= z^2 + a_{m1} z + a_{m2} \\ B_m(z) &= b_0 z + b_1 = B^- B_m^1 \end{aligned}$$

Following the textbook, we see immediately from this that there is no zero cancellation to occur, i.e. $B^+ = 1$, $B^- = B$. Compatibility conditions give that $\deg A_o = \deg A - \deg B^+ - 1 = 1$, i.e. $A_o(z) = z$. The Diophantine equation to solve is

$$AR' + B^- S = A_o A_m$$

where $\deg R' = \deg R = \deg S = 1$. This yields the following set of equations for deciding the coefficients of R and S:

$$\begin{pmatrix} 1 & b_0 & 0 \\ a_1 & b_1 & b_0 \\ a_2 & 0 & b_1 \end{pmatrix} \begin{pmatrix} r_1 \\ s_0 \\ s_1 \end{pmatrix} = \begin{pmatrix} a_{m1} - a_1 \\ a_{m2} - a_2 \\ 0 \end{pmatrix} \tag{5}$$

T is then chosen as $T = A_o B_m^1 = q$. Since the parameters of the true system are not known a priori, we can use the RLS algorithm to estimate them, and use the estimates to iteratively update our controller parameters given by the solution to (5).

6. The system

$$G(q) = \frac{q+3}{q(q+0.6)}$$

is to be controlled using iterative learning control (ILC) according to the block diagram in Figure 2. The control signal for the ILC controller is given by

$$u_{k+1}(t) = u_k(t) + L(q)e_k(t)$$

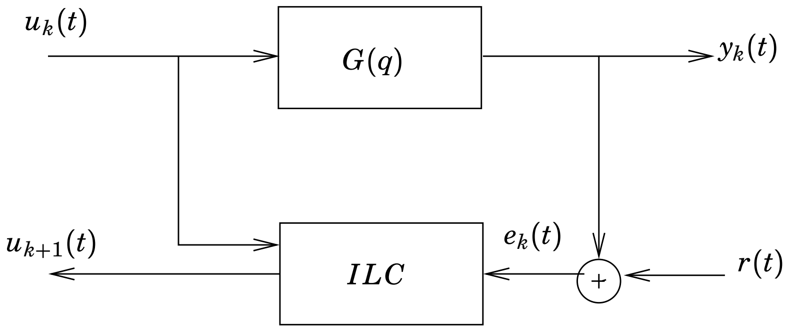
a. The magnitude of the Bode plot for $1 - L(q)G(q)$ is given in Figure 3 for $L(q) = 0.05$ and $L(q) = 0.08q^2$. For which of the two $L(q)$ is the ILC algorithm convergent? Motivate your answer. (1 p)

b. Show that the tracking error fulfills the recursive equation

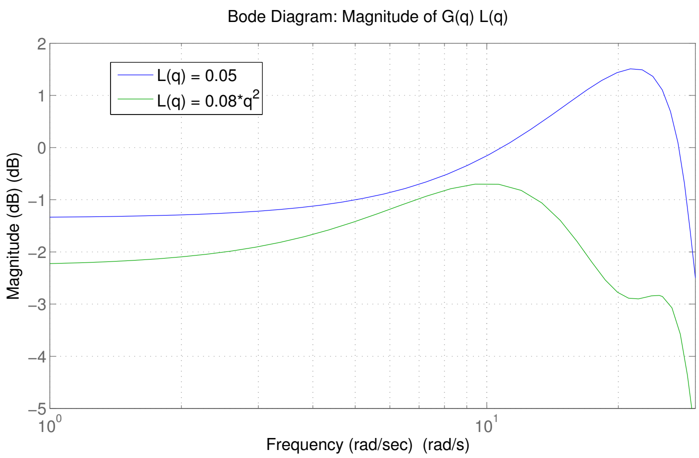
$$e_k(t) = [(1 - Q(q))(1 - T_c(q))]y_d(t) + [Q(q)(1 - L(q)T_c(q))]e_{k-1}(t).$$

What happens if $Q = 1$? What happens if $Q \neq 1$?. (2 p)

Solution



Figur 2 AN ILC feedback system.



Figur 3 Bode Magnitude Plot for $1 - L(q)G(q)$ with $L(q) = 0.05$ and $L(q) = 0.08q^2$.

a. The error between iteration is obtained as:

$$\begin{aligned} e_{k+1}(t) &= r(t) - y_{k+1}(t) = r(t) - G(q)u_{k+1}(t) \\ &= r(t) - G(q)u_k(t) - G(q)L(q)e_k(t) = (1 - G(q)L(q))e_k(t) \end{aligned}$$

The condition for the error not to grow is:

$$|1 - G(e^{i\omega})L(e^{i\omega})| < 1, \quad \forall \omega \in [-\pi, \pi]$$

From the Bode magnitude plot we see that this is the case for $L(q) = 0.08q^2$.

b.

$$
\begin{aligned}
y_k(t) &= T_c(q)y_d(t) + T_c(q)u_k(t) \\
e_k(t) &= y_d(t) - y_k(t) \\
u_k(t) &= Q(q)[u_{k-1}(t) + L(q)e_{k-1}(t)] \\
\implies e_k(t) &= (1 - T_c(q))y_d(t) - T_c(q)u_k(t) \\
&= (1 - Tc(q))y_d(t) - T_c(q)Q(q)[u_{k-1}(t) + L(q)e_{k-1}(t)] \\
&= (1 - T_c(q))y_d(t) - Q(q()T_c(q)u_{k-1}(t) - Q(q)T_c(q)L(q)e_{k-1}(t) \\
&= (1 - Tc(q))y_d(t) - Q(q()T_c(q)u_{k-1}(t) \\
&\quad -Q(q)T_c(q)L(q)e_{k-1}(t) - Q(q)y_d(t) + Q(q)y_d(t) \\
&= (1 - Q(q))(1 - Tc(q))y_d + Q(q)y_d - Q(q)y_{k-1} - Q(q)T_c(q)L(q)e_{k-1} \\
&= (1 - Q(q))(1 - Tc(q))y_d + Q(q)(1 - Tc(q)L(q))e_{k-1}
\end{aligned}
$$

For $Q = 1$ there will be no residual error, for any $Q \neq 1$ there will be a residual error assuming that $T_c(q) \neq 1$. If $T_c(q) = 1$ ILC would be pointless since the closed-loop system would be easily inverted and a perfect u could be found without iterating.

7. In this problem you are to design a two degrees-of-freedom controller on the form

$$u(t) = -k_1(t)y(t) + k_2(t)u_c(t)$$

for the system

$$\dot{y}(t) = ku(t)$$

so that the response follows the model system

$$\dot{y}_m(t) = -ay_m(t) + bu_c(t)\ ,\ a > 0$$

However, it turns out that the gain parameter k is unknown. Your colleague has come up with a parameter update law that produces good result in simulation, but needs your help to give a theoretical justification for it. Verify that your colleagues parameter update law given by

$$
\begin{aligned}
\dot{k}_1 &= e(t)y(t) \\
\dot{k}_2 &= -e(t)u_c(t)
\end{aligned}
$$

indeed guarantees that the error $e(t) = y(t) - y_m(t)$ goes to zero. You may use the following Lyapunov candidate (or any other of your choice):

$$V(x) = V(e, k_1, k_2) = \frac{1}{2}\left(e(t)^2 + \frac{1}{k}(kk_1(t) - a)^2 + \frac{1}{k}(kk_2(t) - b)^2\right)$$

(3 p)

Solution

First we validate that the candidate indeed is a Lyapunov function: $V(0) = 0$, $V(x) > 0, \ \forall x \neq 0$. We then calculate the time derivative:

$$\begin{aligned}\dot{V} &= e\dot{e} + (kk_1 - a)\dot{k}_1 + (kk_2 - b)\dot{k}_2 \\ &= e(-kk_1 y + kk_2 u_c + ay_m - bu_c) + (kk_1 - a)ey - (kk_2 - b)eu_c \\ &= eay_m - eay = -ae^2 < 0 \ , \ \forall e \neq 0\end{aligned}$$

This guarantees that the error e converges to zero.