Predictive Control – Exercise Session 3

Optimal Prediction, Optimal Estimation, Kalman Filter

1 a. The characteristic polynomial of the observer is given by

$$\det(zI - A + KC) = \det \begin{pmatrix} z - 0.78 & k_1 \\ -0.22 & z - 1 + k_2 \end{pmatrix}$$
$$= z^2 + (-1.78 + k_2)z + 0.78 - 0.78k_2 + 0.22k_1$$

The desired characteristic polynomial is z^2. Equating the coefficients, we get

$$\begin{cases} -1.78 + k_2 = 0 \\ 0.78 + 0.22k_1 - 0.78k_2 = 0 \end{cases}$$
$$\Rightarrow \quad K = \begin{pmatrix} 2.77 & 1.78 \end{pmatrix}^T$$

b. The characteristic polynomial of the observer is given by

$$\det(zI - A + KCA) = \det \begin{pmatrix} z - 0.78 + 0.22k_1 & k_1 \\ -0.22 + 0.22k_2 & z - 1 + k_2 \end{pmatrix}$$
$$= z^2 + (-1.78 + 0.22k_1 + k_2)z + 0.78 - 0.78k_2$$

The desired characteristic polynomial is z^2. Equating the coefficients, we get

$$\begin{cases} -1.78 + 0.22k_1 + k_2 = 0 \\ 0.78 - 0.78k_2 = 0 \end{cases}$$
$$\Rightarrow \quad K = \begin{pmatrix} 3.55 & 1 \end{pmatrix}^T$$

2 a. Using the Kalman filter algorithm outlined in Table 7.1 in *Predictive and Adaptive Control* gives:

$$\begin{cases} \hat{x}_{k+1|k} = 0.5\hat{x}_{k|k-1} + K_k(y_k - \hat{x}_{k|k-1}), \quad \hat{x}_{0|-1} = 0 \\ K_k = \dfrac{0.5P_k}{r_2 + P_k} \\ P_{k+1} = 0.25P_k + r_1 - \dfrac{0.25P_k^2}{r_2 + P_k}, P_0 = r_0 \end{cases}$$

b. The stationary variance is given by the positive solution to

$$P^2 + (0.75r_2 - r_1)P = r_1 r_2$$

The stationary gain is then given by

$$K = \frac{0.5P}{r_2 + P}$$

c. In the case $r_1 \gg r_2$, we get $P = r_1$, $K = 0.5$, and $A - KC = 0$. The filter equations reduce to

$$\hat{x}_{k+1|k} = 0.5 y_k$$

Since there is very little measurement noise compared to process noise, the filter can rely on the measurements and gets a deadbeat response.

In the case $r_1 \ll r_2$, we get $P = 1.33 r_1$, $K = 0$, and $A - KC = 0.5$. The filter equations reduce to

$$\hat{x}_{k+1|k} = 0.5 \hat{x}_{k|k-1}$$

Since there is very much measurement noise compared to process noise, the filter relies entirely on prediction and has the same pole as the system.

3. The system is:

$$y_k = \frac{C(z^{-1})}{A(z^{-1})} w_k$$

where:

$$C(z^{-1}) = 1 + 0.7z^{-1}$$
$$A(z^{-1}) = 1 - 1.5z^{-1} + 0.9z^{-2}$$

a. We are intereseted in finding a one step ahead prediction of the output, y_{k+1}. A prediction of y_{k+1} could be obtained by simply ignoring the noise: $\hat{y}_{k+1} = 1.5y_k - 0.9y_{k-1}$. However this is not the optimal prediction. Information about the noise w_k is available in the measured data available at time k. Consider the diophantine equation:

$$C = AF + z^{-1}G \tag{1}$$

where F and G are polynomials to be determined. The output can be written as:

$$y_{k+1} = F w_{k+1} + \frac{G}{A} w_k$$

The first term involves future signals, unkown at time k, while the second term contains only signals available at time k. The optimal predictor is then given by:

$$\hat{y}_{k+1} = \frac{G}{A} w_k = \frac{G}{C} y_k$$

Choosing F of order $d - 1$ where d is the delay of the system and G of order $n - 1$ where n is the order of the system allows the coefficients to be calculated by comparing powers of z^{-1} in (1). This gives:

$$F = f_0 = 1$$
$$G = g_0 + g_1 z^{-1} = 2.2 - 0.9z^{-1}$$

The optimal predictor is then:

$$\hat{y}_{k+1} = \frac{2.2 - 0.9z^{-1}}{1 + 0.7z^{-1}} y_k$$

The variance of the prediction is:

$$\mathcal{E}\{(\hat{y}_{k+1} - y_{k+1})^2\} = \mathcal{E}\{(Fw_{k+1})^2\} = f_0^2\sigma_w^2 = \sigma_w^2$$

b. For the two step ahead predictor, the Diophantine equation is:

$$C = AF + z^{-2}G \tag{2}$$

The polynomial F is now of first order. By comparing coefficients in the same manner as in (a), we get:

$$F = f_0 + f_1z^{-1} = 1 + 2.2z^{-1}$$
$$G = g_0 + g_1z^{-1} = 2.4 - 1.98z^{-1}$$

The optimal predictor is then:

$$\hat{y}_{k+2} = \frac{2.4 - 1.98z^{-1}}{1 + 0.7z^{-1}} y_k$$

The variance of the prediction is:

$$\mathcal{E}\{(\hat{y}_{k+2} - y_{k+2})^2\} = \mathcal{E}\{(Fw_{k+2})^2\} = (f_0^2 + f_1^2)\sigma_w^2 = 5.84\sigma_w^2$$

As one might expect, the variance of the two step ahead predictor is much higher than the one step ahead version.

4. Minimum variance control involves finding a control law which minimizes the variance of the output. In this example the system is given by: The system is:

$$y_{k+1} = \frac{B(z^{-1})}{A(z^{-1})}u_k + \frac{C(z^{-1})}{A(z^{-1})}w_{k+1}$$

where:

$$A(z^{-1}) = 1 - 1.5z^{-1} + 0.9z^{-2}$$
$$B(z^{-1}) = 1 + 0.9z^{-1}$$
$$C(z^{-1}) = 1 + 0.7z^{-1}$$

a. The one step ahead minimum variance controller is related to the one step ahead predictor in the previous question. The predictor was obtained by solving the Diophantine equation:

$$C = AF + z^{-1}G \tag{3}$$

giving:

$$F = f_0 = 1$$
$$G = g_0 + g_1 z^{-1} = 2.2 - 0.9z^{-1}$$

Inserting (3) into the system dynamics gives:

$$y_{k+1} = F(z^{-1})w_{k+1} + \frac{B(z^{-1})}{A(z^{-1})}u_k + \frac{G(z^{-1})}{A(z^{-1})}w_k \tag{4}$$

The noise term w_k can be obtained from measured data at time k:

$$w_k = \frac{A(z^{-1})}{C(z^{-1})}y_k - \frac{B(z^{-1})}{C(z^{-1})}z^{-1}u_k$$

Introducing this in (4) and rearranging gives:

$$y_{k+1} = F(z^{-1})w_{k+1} + \frac{B(z^{-1})F(z^{-1})}{C(z^{-1})}u_k + \frac{G(z^{-1})}{C(z^{-1})}y_k \tag{5}$$

Clearly, choosing u_k such that the final two terms on the right hand side of (5) cancel out gives the minimum value of the variance of y_{k+1}. Thus the minimum variance controller is given by:

$$u_k = -\frac{G(z^{-1})}{B(z^{-1})F(z^{-1})}y_k = -\frac{2.2 - 0.9z^{-1}}{1 + 0.9z^{-1}}y_k$$

The output variance is then:

$$\mathcal{E}\{(y_{k+1})^2\} = \mathcal{E}\{(Fw_{k+1})^2\} = f_0^2\sigma_w^2 = \sigma_w^2$$

b. The two step ahead minimum variance controller is obtained in a similar way as the one step ahead version. It is given by:

$$u_k = -\frac{G(z^{-1})}{B(z^{-1})F(z^{-1})}y_k = -\frac{2.4 - 1.98z^{-1}}{(1 + 0.9z^{-1})(1 + 2.2z^{-1})}y_k$$

with output variance:

$$\mathcal{E}\{(y_{k+2})^2\} = \mathcal{E}\{(Fw_{k+2})^2\} = (f_0^2 + f_1^2)\sigma_w^2 = 5.84\sigma_w^2$$

c. Without zero cancellation, the Diophantine equation to be solved is:

$$C = AR + z^{-1}BS \tag{6}$$

and the controller is given by:

$$u_k = -\frac{S(z^{-1})}{R(z^{-1})}y_k$$

In the previous parts we had $S = G$ and $R = BF$, but in this case R no longer contains B. Solving (6) gives:

$$R = r_0 + r_1 z^{-1} = 1 + 0.8471 z^{-1}$$
$$S = s_0 + s_1 z^{-1} = 1.3529 - 0.8471 z^{-1}$$

The closed loop is given by:

$$y_{k+1} = \frac{CR}{AR + BS} w_k = R w_k = (1 + 0.8471 z^{-1}) w_{k+1}$$

which gives an output variance:

$$\mathcal{E}\{(y_{k+1})^2\} = \mathcal{E}\{(F w_{k+1})^2\} = (f_0^2 + f_1^2)\sigma_w^2 = 1.7176\sigma_w^2$$

This variance is higher than in the case of zero cancellation in (a).