Sampling

We start out with a continuous-time system on state-space form:

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1}$$
$$y(t) = Cx(t). \tag{2}$$

If the system is given as a continuous time transfer function, you can easily rewrite it as one of several state space representations. (Search for canonical state-space forms if you get stuck.) To solve (1) we multiply both sides with the integrating factor e^{-At} to obtain.

$$e^{-At}\dot{x}(t) = e^{-At}Ax(t) + e^{-At}Bu(t), \Leftrightarrow \tag{3}$$
$$e^{-At}\dot{x}(t) - e^{-At}Ax(t) = \frac{d}{dt}\left[e^{-At}x(t)\right] = e^{-At}Bu(t). \tag{4}$$

Hence

$$e^{-At}x(t) = e^{-At_k}x(t_k) + \int_{t_k}^{t} e^{-A\tau}Bu(\tau)d\tau, \tag{5}$$

by the Fundamental theorem of Calculus. Zero order hold sampling implies

$$u(\tau) = u(t_k),\ \forall\ \tau\ \in [t_k, t_k + h], \tag{6}$$

and hence (5) can be rewritten

$$x(t_k + h) = e^{A(t_k+h-t_k)}x(t_k) + \int_{t_k}^{t_k+h} e^{A(t_k+h-\tau)}Bu(t_k)d\tau. \tag{7}$$

Introducing

$$h = t_{k+1} - t_k \tag{8}$$
$$s = \tau - t_k \tag{9}$$

we can rewrite (7)

$$x(t_k + h) = e^{Ah}x(t_k) + \int_0^h e^{As}Bu(t_k)ds. \tag{10}$$

Finally we introduce the constant matrices

$$\Phi_h = e^{Ah}, \tag{11}$$
$$\Gamma_h = \int_0^h e^{As}dsB, \tag{12}$$

and arrive at the sampled system

$$x(t_k + h) = \Phi_h x(t_k) + \Gamma_h u(t_k), \tag{13}$$
$$y(t_k) = Cx(t_k). \tag{14}$$