Cheat Sheet – FRTN50

Convex sets

- $C \subseteq \mathbb{R}^n$ is convex if for all $x, y \in C$ and $\theta \in [0,1]$: $\theta x + (1-\theta)y \in C$.
- Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be a convex function, then $C = \{x \in \mathbb{R}^n : f(x) \leq 0\}$ is a convex set.

Convex functions

- $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is convex if for all $x, y \in \mathbb{R}^n$ and $\theta \in [0,1]$: $f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y)$
- Differentiable $f : \mathbb{R}^n \to \mathbb{R}$ is convex if for all $x, y \in \mathbb{R}^n$: $f(y) \geq f(x) + \nabla f(x)^T(y-x)$
- $f = h \circ g : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is convex for $g : \mathbb{R}^n \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ if one of the following holds:
  - h is convex and nondecreasing and g is convex
  - h is convex and nonincreasing and g is concave
  - h is convex and g is affine

Subgradients

- Subgradient to $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ at x is any vector $s \in \mathbb{R}^n$ such that for all $y \in \mathbb{R}^n$: $f(y) \geq f(x) + s^T(y-x)$
- Set of subgradients at x, denoted by $\partial f(x)$, is called subdifferential at x and operator ∂f called subdifferential
- Fermat's rule: $x \in \mathbb{R}^n$ minimizes $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ if and only if $0 \in \partial f(x)$
- $f, g : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ are closed convex and constraint qualification holds: $\partial(f+g) = \partial f + \partial g$
- $g : \mathbb{R}^m \to \mathbb{R} \cup \{\infty\}$ is closed convex, $L \in \mathbb{R}^{m \times n}$, and constraint qualification holds: $\partial(g \circ L)(x) = L^T \partial g(Lx)$

Conjugate functions

- Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$, then conjugate $f^* : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is defined as $f^*(s) = \sup_x(s^T x - f(x))$
- Fenchel-Young's inequality: $f(x) + f^*(s) \geq s^T x$ for all $x, s \in \mathbb{R}^n$
- Equivalence due to Fenchel Young: $f(x) + f^*(s) = s^T x$ if and only if $s \in \partial f(x)$
- Suppose $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is closed convex, then $f^{**} = f$

Duality

- Assumptions: $f : \mathbb{R}^n \to \mathbb{R}^n \cup \{\infty\}$, $g : \mathbb{R}^m \to \mathbb{R} \cup \{\infty\}$ closed convex, $L \in \mathbb{R}^{m \times n}$, constraint qualification holds
- Given assumptions: x solves $\text{minimize}_x(f(Lx) + g(x))$ if and only if $0 \in L^T \partial f(Lx) + \partial g(x)$
- Let $\mu \in \partial f(Lx)$ to arrive at dual problem: $\text{minimize}_\mu(f^*(\mu) + g^*(-L^T\mu))$

Strong convexity and smoothness

- $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is σ-strongly convex with $\sigma > 0$ if $f - \frac{\sigma}{2}\|\cdot\|_2^2$ is convex
- σ-strongly convex $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ satisfies for all $s \in \partial f(x)$ and $y \in \mathbb{R}^n$: $f(y) \geq f(x) + s^T(y-x) + \frac{\sigma}{2}\|x-y\|_2^2$
- Differentiable $f : \mathbb{R}^n \to \mathbb{R}$ is β-smooth with $\beta \geq 0$ if ∇f is β-Lipschitz continuous
- β-smooth $f : \mathbb{R}^n \to \mathbb{R}$ satisfies descent lemma, for all x, y: $f(y) \leq f(x) + \nabla f(x)^T(y-x) + \frac{\beta}{2}\|x-y\|_2^2$
- $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is σ-strongly convex if and only if f^* is σ^{-1}-smooth and convex

Proximal gradient method

- Proximal operator of $g : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is $\text{prox}_{\gamma g}(z) = \text{argmin}_x(g(x) + \frac{1}{2\gamma}\|x-z\|_2^2)$
- Proximal gradient iteration: $x_{k+1} = \text{prox}_{\gamma_k g}(x_k - \gamma_k \nabla f(x_k))$