Convex Sets

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Today's lecture

Motivation and context

- What is optimization?
- Why optimization?
- Convex vs nonconvex optimization

Convex sets

- Definition
- Examples of convex sets
- Separating and supporting hyperplanes

What is optimization?

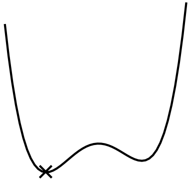
- Find point $x \in \mathbb{R}^n$ that minimizes a function $f : \mathbb{R}^n \to \mathbb{R}$:

$$\underset{x}{\text{minimize}}\, f(x)$$

- Can also require x to belong to a set $S \subset \mathbb{R}^n$:

$$\underset{x \in S}{\text{minimize}}\, f(x)$$

- Example:



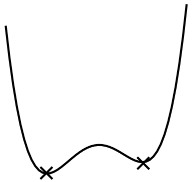
Why optimization?

- Many engineering problems can be modeled using optimization
  - **Supervised learning**
  - Optimal control
  - Signal reconstruction
  - Portfolio selection
  - Image classifiction
  - Circuit design
  - Estimation
  - ...
- Results in "optimal":
  - Model
  - Decision
  - Performance
  - Design
  - Estimate
  - ...

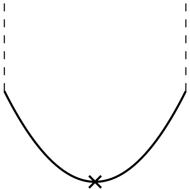
  w.r.t. optimization problem model
- Different question: How good is the model?

Convex vs nonconvex optimization

- Convex optimization if set and function are convex
- Otherwise nonconvex optimization problem
- Why convexity?: Local minima are global minima
- Why go nonconvex?: Richer modeling capabilities



nonconvex function



convex function

- If convex modeling enough, use it, otherwise try nonconvex

Convex Sets

Learning goals

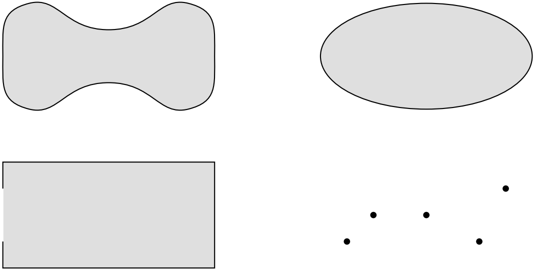
- Know convex set definition
- Understand intersection, union, and convex hull
- Able to decide if set is convex based on
  - Graphical representation of set
  - Mathematical definition of set
- Understand supporting and separating hyperplanes

Convex sets

- A set C is convex if for every $x, y \in C$ and $\theta \in [0, 1]$:

$$\theta x + (1 - \theta) y \in C$$

- "Every line segment that connect any two points in C is in C"



- Will assume that all sets are nonempty and closed

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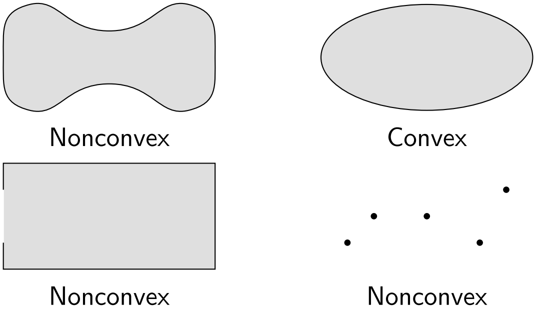
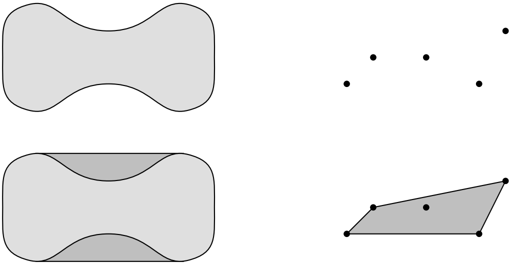


- Will assume that all sets are nonempty and closed

Convex combination and convex hull

Convex hull ($\mathrm{conv} S$) of S is smallest convex set that contains S:



Mathematical construction:

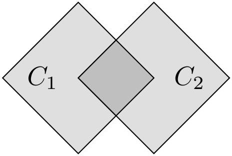
- Convex combinations of $x_1, \ldots, x_k$ are all points x of the form

$$x = \theta_1 x_1 + \theta_2 x_2 + \ldots + \theta_k x_k$$

  where $\theta_1 + \ldots + \theta_k = 1$ and $\theta_i \geq 0$
- Convex hull: set of all convex combinations of points in S

Intersection and union

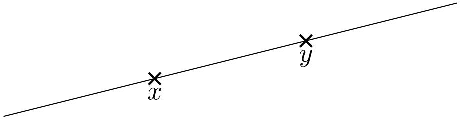
- Intersection $C = C_1 \cap C_2$ means $x \in C$ if $x \in C_1$ **and** $x \in C_2$
- Union $C = C_1 \cup C_2$ means $x \in C$ if $x \in C_1$ **or** $x \in C_2$



- Intersection of two convex sets is convex
- Union of two convex sets need not be convex

Affine sets

- Take any two points $x, y \in V$: V is affine if full line in V:



  Lines and planes are affine sets

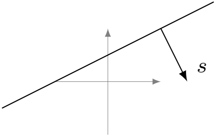
- Definition: A set V is affine if for every $x, y \in V$ and $\alpha \in \mathbb{R}$:

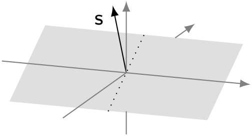
$$\alpha x + (1-\alpha) y \in V \tag{1}$$

  hence convex this holds in particular for $\alpha \in [0,1]$

Affine hyperplanes

- Affine hyperplanes in $\mathbb{R}^n$ are affine sets that cut $\mathbb{R}^n$ in two halves





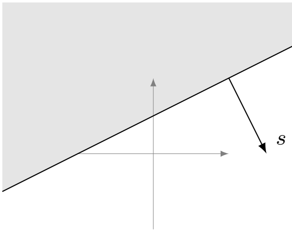
- Dimension of affine hyperplane in $\mathbb{R}^n$ is $n-1$ (If $s \neq 0$)
- All affine sets in $\mathbb{R}^n$ of dimension $n-1$ are hyperplanes
- Mathematical definition:

$$h_{s,r} := \{x \in \mathbb{R}^n : s^T x = r\}$$

  where $s \in \mathbb{R}^n$ and $r \in \mathbb{R}$, i.e., defined by one *affine function*
- Vector s is called normal to hyperplane

Halfspaces

- A halfspace is one of the halves constructed by a hyperplane



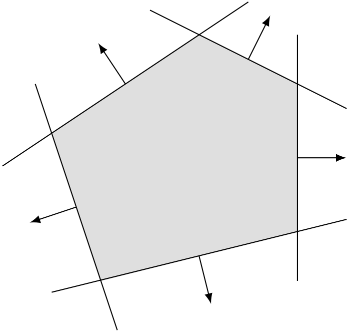
- Mathematical definition:

$$H_{r,s} = \{x \in \mathbb{R}^n : s^T x \leq r\}$$

- Halfspaces are convex, and vector s is called normal to halfspace

Polytopes

- A *polytope* is intersection of halfspaces and hyperplanes



- Mathematical representation:

$$C = \{x \in \mathbb{R}^n : s_i^T x \le r_i \text{ for } i \in \{1, \ldots, m\} \text{ and } s_i^T x = r_i \text{ for } i \in \{m+1, ..., p\}\}$$

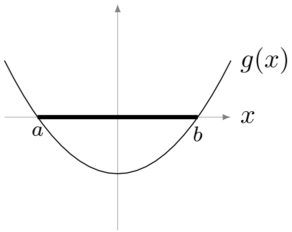
- Polytopes convex since intersection of convex sets

Set defined by convex function

- Suppose that $g : \mathbb{R}^n \to \mathbb{R}$ is a convex function
- The sublevel set of g:

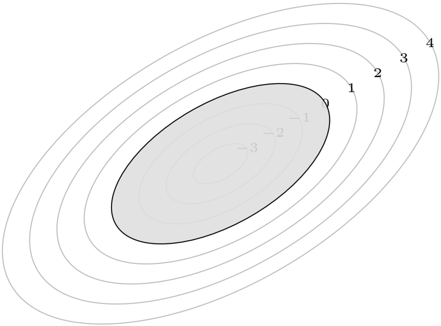
$$C = \{x \in \mathbb{R}^n : g(x) \leq 0\}$$

  is a convex set
- Example: construction giving 1D interval $[a, b]$



Examples

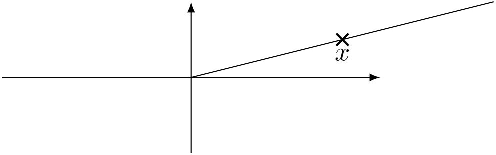
- Example: Levelsets of convex quadratic function



- Norm balls $\{x \in \mathbb{R}^n : \|x\| - r \leq 0\}$
- Ellipsoid $\{x \in \mathbb{R}^n : \frac{1}{2}x^T Px + q^T x + r \leq 0\}$, P positive definite

Cones

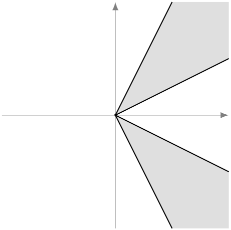
- Take any point $x \in K$: K is a cone if full ray in K



- Definition: A set K is a cone if for all $x \in K$ and $\alpha \geq 0$: $\alpha x \in K$

Cones – Examples

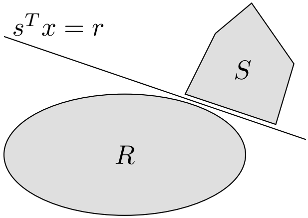
- A nonconvex cone



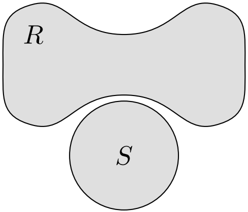
- Convex cones:
  - Linear subspaces $\{x \in \mathbb{R}^n : Ax = 0\}$ (but not affine subspaces)
  - Halfspaces based on linear (not affine) hyperplanes $\{x : s^T x \leq 0\}$
  - Positive semi-definite matrices
    $\{X \in \mathbb{R}^{n \times n} : X$ symmetric and $x^T X x \geq 0$ for all $x \in \mathbb{R}^n\}$
  - Nonnegative orthant $\{x \in \mathbb{R}^n : x \geq 0\}$

Separating hyperplane theorem

- Suppose that $R, S \subseteq \mathbb{R}^n$ are two non-intersecting convex sets
- Then there exists hyperplane with S and R in opposite halves



Example



Counter-example
R nonconvex

- Mathematical formulation: There exists $s \neq 0$ and r such that

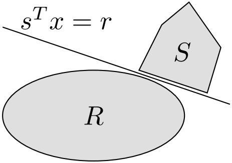
$$s^T x \le r \qquad \text{for all } x \in R$$

$$s^T x \ge r \qquad \text{for all } x \in S$$

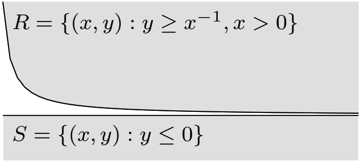
- The hyperplane $\{x : s^T x = r\}$ is called *separating hyperplane*

A strictly separating hyperplane theorem

- Suppose that $R, S \subseteq \mathbb{R}^n$ are non-intersecting closed and convex sets and that one of them is compact (closed and bounded)
- Then there exists hyperplane with strict separation



Example



Counter example
R, S not compact

- Mathematical formulation: There exists $s \neq 0$ and r such that

$$s^T x < r \qquad \text{for all } x \in R$$

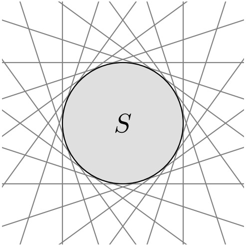
$$s^T x > r \qquad \text{for all } x \in S$$

Consequence – S is intersection of halfspaces

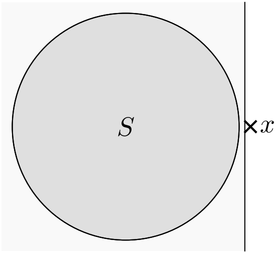
$\boxed{\text{a closed convex set } S \text{ is the intersection of all halfspaces that contain it}}$

proof:

- let H be the intersection of all halfspaces containing S
- $\Rightarrow$: obviously $x \in S \Rightarrow x \in H$
- $\Leftarrow$: assume $x \notin S$, since S closed and convex and x compact (a point), there exists a strictly separating hyperplane, i.e., $x \notin H$:



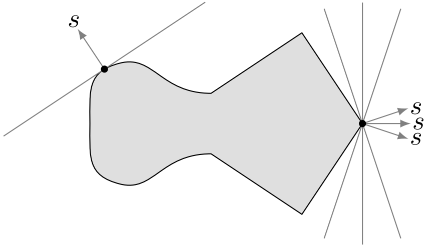
$\Rightarrow$



$\Leftarrow$

Supporting hyperplanes

- Supporting hyperplanes touch set and have full set on one side:



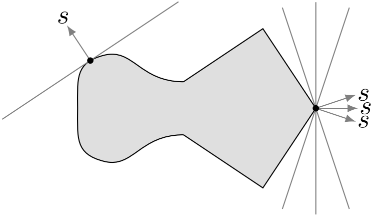
- We call the halfspace that contains the set *supporting halfspace*
- s is called *normal vector* to S at x
- Definition: Hyperplane $\{y : s^T y = r\}$ supports S at $x \in$ bd S if

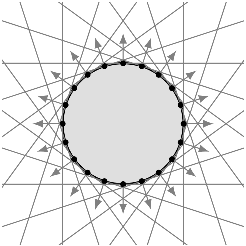
$$s^T y \leq r \text{ for all } y \in S \qquad \text{and} \qquad s^T x = r$$

Supporting hyperplane theorem

Let S be a nonempty convex set and let $x \in \mathrm{bd}(S)$. Then there exists a supporting hyperplane to S at x.

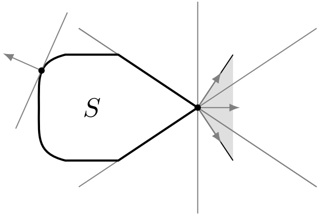
- Does not exist for all point on boundary for nonconvex sets
- Many supporting hyperplanes exist for points of nonsmoothness





Normal cone operator

- Normal cone operator contains normals to supporting hyperplanes



- Defined also for points not on boundary
  - For $x \in S$: $0 \in N_S(x)$
  - For $x \in \mathrm{int}\ S$: the normal cone $N_S(x) = 0$
- Definition: The normal cone operator to a set S is

$$N_S(x) = \begin{cases} \{s : s^T(y-x) \leq 0 \text{ for all } y \in S\} & \text{if } x \in S \\ \emptyset & \text{else} \end{cases}$$

i.e., vectors that form obtuse angle between s and all $y - x$, $y \in S$