Fundamental Limitations in MIMO Systems

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Outline

Singular Values

- The singular values of a matrix A
  - $\sigma_i = \sqrt{\lambda_i}$, where λ_i eigenvalues to A^*A.
  - Largest s.v. denoted $\bar{\sigma}(A)$
  - Smallest s.v. denoted $\underline{\sigma}(A)$
  - For $y = Ax$ we have that $\underline{\sigma}(A) \leq \frac{|y|}{|x|} \leq \bar{\sigma}(A)$
- The gain is limited by the singular values
- The actual gain depends on the *direction* of the input vector
- Example: $Y(i\omega) = G(i\omega)U(i\omega)$
  - If $U(i\omega)$ parallel with the eigenvector connected to the largest eigenvalue of $G^*(i\omega)G(i\omega)$, then the gain is $\bar{\sigma}(G)$.

Pole and zero directions

Zero directions

- $G(z)u_z = 0$, u_z input direction
- $y_z^H G(z) = 0$, y_z output direction
- Normalized so $||u_z||_2 = 1$, $||y_z||_2 = 1$

Pole directions

- $G(p)u_p = \infty$, u_p input direction
- $y_p^H G(p) = \infty$, y_p output direction
- Normalized so $||u_p||_2 = 1$, $||y_p||_2 = 1$

Sensitivity functions

- $S = (I + GK)^{-1}$, Sensitivity function from output disturbance to output.
- $T = (I + GK)^{-1}GK$, Complementary sensitivity function, from reference to output.
- $S + T = I$
  - $|1 - \bar{\sigma}(S)| \leq \bar{\sigma}(T) \leq 1 + \bar{\sigma}(S)$
  - $|1 - \bar{\sigma}(T)| \leq \bar{\sigma}(S) \leq 1 + \bar{\sigma}(T)$
  - As in SISO both $\bar{\sigma}(S)$ and $\bar{\sigma}(T)$ can not be small simultaneously.
  - It is also clear that $\bar{\sigma}(S)$ is large ($\gg 1$) iff $\bar{\sigma}(T)$ is large.

Bode's integral theorem

SISO

- $\int_0^\infty \log|S(i\omega)|d\omega = \pi \sum_{i=1}^{M} Re(p_i)$
- The area of $|S(i\omega)|$ above and below 1 is the same if no unstable poles. If unstable poles p_i the area above 1 is larger.

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MIMO

- $\int_0^\infty \log |\det S(i\omega)| d\omega = \pi \sum_{i=1}^{M} Re(p_i)$
- $|\det S| = \sigma_1(S) \cdots \sigma_m(S)$
- $\sum_{k=1}^{m} \int_0^\infty \log \sigma_k(S(i\omega)) d\omega = \pi \sum_{i=1}^{M} Re(p_i)$
- $\int_0^\infty \log \bar{\sigma}(S(i\omega)) d\omega \geq \frac{\pi}{m} \sum_{i=1}^{M} Re(p_i)$
- Not as clear conclusions as in the SISO case.

Interpolation Constraints

RHP poles & zeros give *interpolation constraints*.

Necessary for internal stability:

SISO:

RHP Zero

$$T(z) = 0, \quad S(z) = 1$$

for every RHP zero z in $G(s)$

RHP Pole

$$S(p) = 0, \quad T(p) = 1$$

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MIMO:

RHP Zero

$$y_z^H T(z) = 0, \quad y_z^H S(z) = y_z^H$$

for every RHP zero z in $G(s)$

RHP Pole

$$S(p)y_p = 0, \quad T(p)y_p = yp$$

for every RHP pole p in $G(s)$

Conditions on Weighted S and T

From Interp. Constr. and Max. Mod. Theorem we have
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Sensitivity Function:

$$||W_S S||_\infty = \sup_\omega |W_S(i\omega)|\bar{\sigma}(S(i\omega)) \geq |W_S(z)|$$

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Complementary Sensitivity Function:

$$||W_T T||_\infty = \sup_\omega |W_T(i\omega)|\bar{\sigma}(T(i\omega)) \geq |W_T(p)|$$

for every RHP pole p

Note: $|\cdot|$ in SISO $\rightarrow \bar{\sigma}(\cdot)$ in MIMO

Limitations Imposed By RHP-Zeros

As in SISO, RHP-zeros in MIMO systems set **upper bandwidth-limit**

Consider specification $||W_S S||_\infty \leq 1$, which implies:

$$\bar{\sigma}(S(i\omega)) \leq |W_S^{-1}(i\omega)|$$

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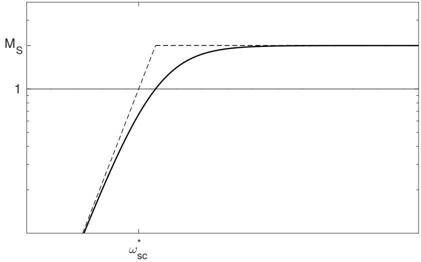
Common to choose W_S^{-1} on form:

$$W_S^{-1} = \frac{M_s s}{s+\omega_{sc}^* M_S}$$



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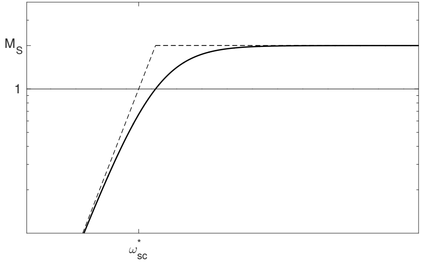

$$|W_S(z)| \leq ||W_S S||_\infty \leq 1$$
$$\implies \omega_{sc}^* \leq (1 - 1/M_S)z$$

$$M_S = 2 \implies \omega_{sc}^* \leq z/2$$

Limitations Imposed By RHP-Zeros

- The well-known upper-bandwidth **rule of thumb** $\omega_{sc}^* \leq z/2$ still holds in MIMO case, but for "worst" direction, i.e direction of $\bar{\sigma}(S(i\omega))$
- The RHP-zero **might not** be a limitation in another output direction!
- We may to some extent in our controller design **choose** the worst direction

Example: RHP-Zero

Consider the following TITO-system:

$$G(s) = \frac{1}{(0.2s+1)(s+1)} \begin{bmatrix} 1 & 1 \\ 1+2s & 2 \end{bmatrix}$$

The system has a RHP-zero z = 0.5, with output direction:

$$y_z = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0.89 \\ -0.45 \end{bmatrix}$$

Example: RHP-Zero

Interpolation constraints give:

$$y_z^T T(z) = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \end{bmatrix} \begin{bmatrix} t_{11}(z) & t_{12}(z) \\ t_{21}(z) & t_{22}(z) \end{bmatrix} = 0$$

i.e:

$$2t_{11}(z) - t_{21}(z) = 0$$
$$2t_{12}(z) - t_{22}(z) = 0$$

We will look at the reference tracking problem $y = Tr$ and examine 3 different specifications for T:

- T_0 : Decoupled design
- T_1 : Perfect tracking of r_1
- T_2 : Perfect tracking of r_2

Example: RHP-Zero

T_0: Decoupled design

To decouple we require $t_{12} = t_{21} = 0$

$$\implies \begin{aligned} t_{11}(z) &= 0 \\ t_{22}(z) &= 0 \end{aligned}$$

i.e the zero $z = 0.5$ **must** show up in both directions, limiting bandwidth to $\approx 0.25 rad/s$ in both channels.

One possible T_0 which fulfills $T_0(0) = I$ is:

$$T_0 = \begin{bmatrix} \frac{-s+z}{s+z} & 0 \\ 0 & \frac{-s+z}{s+z} \end{bmatrix}$$

Example: RHP-Zero

T_1: **Perfect tracking of** r_1

We now specify $t_{11} = 1, t_{12} = 0$

$$\implies \begin{aligned} t_{21}(z) &= 2 \\ t_{22}(z) &= 0 \end{aligned}$$

i.e the zero $z = 0.5$ will still show up in channel 2. With specification $T_1(0) = I$ we get one possible T_1:

$$T_1 = \begin{bmatrix} 1 & 0 \\ \frac{4s}{s+z} & \frac{-s+z}{s+z} \end{bmatrix}$$

Zero moved to affect only y_2, but comes with price of interaction.

Example: RHP-Zero

T_2: **Perfect tracking of** r_2

Similar to tracking of r_1, we specify $t_{22} = 1, t_{21} = 0$

$$\implies t_{11}(z) = 0$$
$$t_{12}(z) = \frac{1}{2}$$

i.e the zero $z = 0.5$ will show up in channel 1. With specification $T_2(0) = I$ we get one possible T_2:

$$T_2 = \begin{bmatrix} \frac{-s+z}{s+z} & \frac{s}{s+z} \\ 0 & 1 \end{bmatrix}$$

Zero moved to affect only y_1, but comes with price of interaction.

Example: RHP-Zero

- Effect of RHP-zero can be moved to different outputs
- However, comes with a price in interaction
- In example, interaction larger in T_1 than T_2.
- Reasonable, since zero-output direction $\begin{bmatrix}0.89 & -0.45\end{bmatrix}^T$ is closer to direction of output 1 than 2. We need to "pay more" to push its effect to output 2.

Limitations Imposed By RHP-Poles

- Similar to SISO, RHP-poles set **lower bandwidth-limit** in also in MIMO.
- Limitations for SISO-systems can be generalized to MIMO by considering "worst direction"
- Rule of thumb still holds, but in $\bar{\sigma}(T)$:

Rule of Thumb: Lower Bandwidth-Limit

$$\omega_{tc}^* \geq 2|p|$$

where ω_{tc}^* is the crossover freq. of $\bar{\sigma}(T)$

RHP Poles Combined with RHP-Zeros

Impossible to achieve good performance if RHP-pole and RHP-zero are closely located to each other.

In case of 1 RHP-pole and 1 RHP-zero it can be shown for a MIMO process:

$$||S||_\infty \geq c, \quad ||T||_\infty \geq c, \quad c = \sqrt{sin^2\phi + \frac{|z+p|^2}{|z-p|^2} cos^2\phi} \geq 1$$

where ϕ is the angle between RHP-zero and RHP-pole

- c large when z and p are aligned and short distance to each other
- General formula for more than 1 RHP-pole and zero exists

Limitations due to Time-Delays

SISO

Limitation on bandwidth $\omega_B \leq 1/\theta$

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SISO

Limitation on bandwidth $\omega_B \leq 1/\theta$

MIMO

- Not so much can be said.
- Lower bound on time delay for output i $\quad \theta_i^{min} = \min_j \theta_{ij}$
- Surprisingly(?) increased time delay can sometimes improve achievable performance.

Limitations due to Time-Delays

SISO

Limitation on bandwidth $\omega_B \leq 1/\theta$

MIMO

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Example

Consider the plant $G(s) = \begin{pmatrix} 1 & 1 \\ e^{-\theta s} & 1 \end{pmatrix}$. With $\theta = 0$ this is singular (not functionally controllable), But the larger θ gets, the easier the control is. To see this you could look at the RGA matrix or the condition number.

Conclusions

- Bode's integral theorem is not as useful in the MIMO case since you either have an expression for all the singular values or an inequality.
- Limitations due to RHP poles and zeros generalize to MIMO but directions become important.
- Not much can be said about constraints due to time-delays, sometimes a larger delay can even improve the achievable performance.

For Further Reading I

T. Glad, L. Ljung
Reglerteori - Flervariabla och olinjära metoder.
Studentlitteratur AB, 2003.

S. Skogestad, I. Postlethwaite
Multivariable Feedback Control - Analysis and Design
John Wiley & Sons, 1996.