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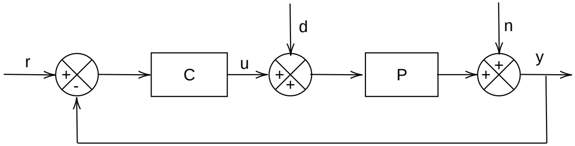
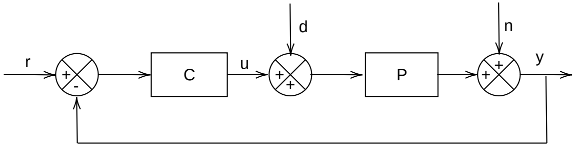
Control System Synthesis - Basics

PHD CLASS - FALL 2020

Summary



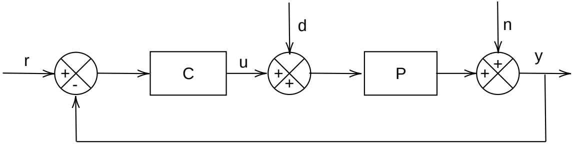
Summary



Requirements:

- Stability
- Reference tracking
- Disturbance rejection
- Noise attenuation
- Robustness to process variations

Summary



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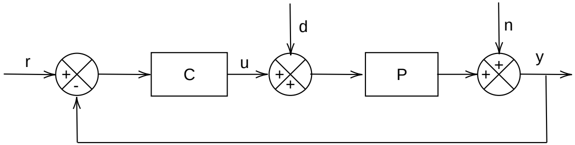
The Gang of Four:

- $S = \frac{1}{1+PC}$
- $T = \frac{PC}{1+PC}$
- $CS = \frac{C}{1+PC}$
- $PS = \frac{P}{1+PC}$

Summary



Two tools

Nyquist plot

- open-loop $L = PC$
- stability criteria
- margins

Bode plot

- closed-loop
- performances of the Gang of Four

Performance specifications
Driving example: Cruise control
Design trade-offs
Expression in the frequency-domain
Loopshaping

Fundamental limitations
System design considerations
Sensitivity minimization
Bode's integral formula
Gain crossover frequency inequality
Internal stability

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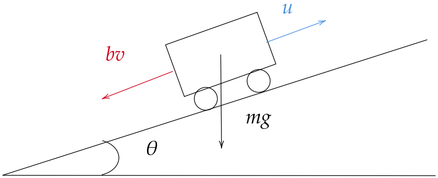
Summary

1. Introduction
2. Fundamentals: problem formulation
   - System representation and feedback basics
   - **Specifications and performance limitations**
3. Design techniques

Content overview

Performance specifications

Driving example: Cruise control

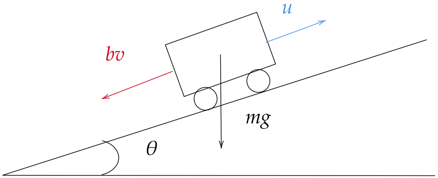


$$m\dot{v} = -bv - mgsin(\theta) + u$$

$$sin(\theta) \sim \theta$$

Performance specifications

Driving example: Cruise control



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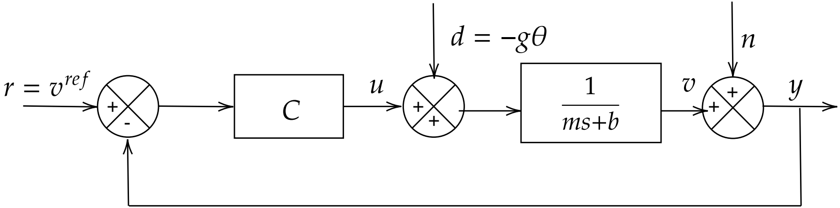


Performance specifications

Driving example: Cruise control

- Follow a reference signal: **the servo problem**
- Act on $T = \frac{V(s)}{R(s)} = \frac{PC}{1+PC}$

In time-domain, we want:

- Rise time < 5s
- Overshoot $< 10\%$
- Steady-state error $< 2\%$

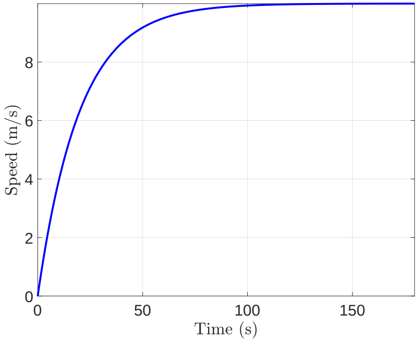


Figure: Step response of the system to $u = 500N$.

Performance specifications

Driving example: Cruise control

- Follow a reference signal: **the servo problem**
- Act on $T = \frac{V(s)}{R(s)} = \frac{PC}{1+PC}$

$$C(s) = k_p$$

$$T(s) = \frac{k_p}{ms + b + k_p}$$

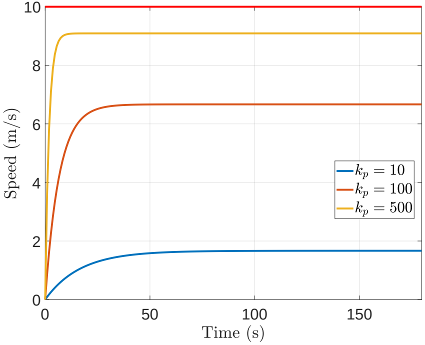
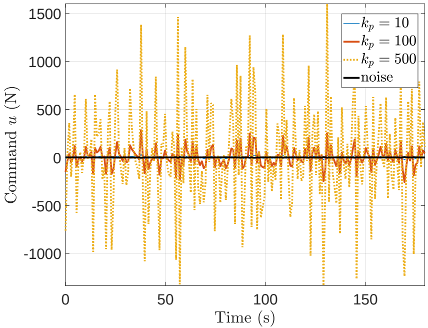


Figure: Step responses of the closed-loop for different k_p.

Performance specifications

Design trade-offs

- Effect on noise sensitivity: $T_{n \to u} = -\frac{C}{1+PC}$



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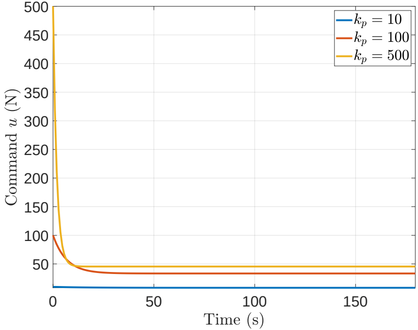
- Effect on the command: $T_{r\to u} = \frac{C}{1+PC} \to$ Limitations of the actuators!



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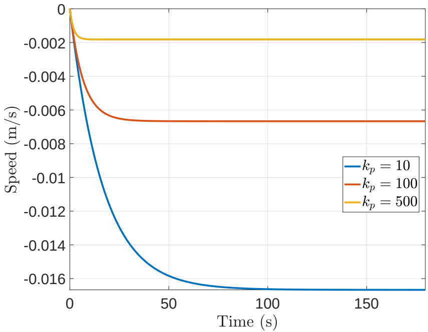
- Effect on disturbance rejection $T_{r\to v} = \frac{P}{1+PC}$



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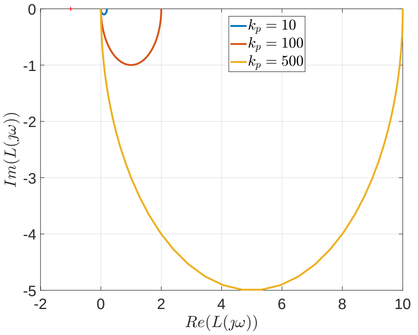
- Effect on robustness



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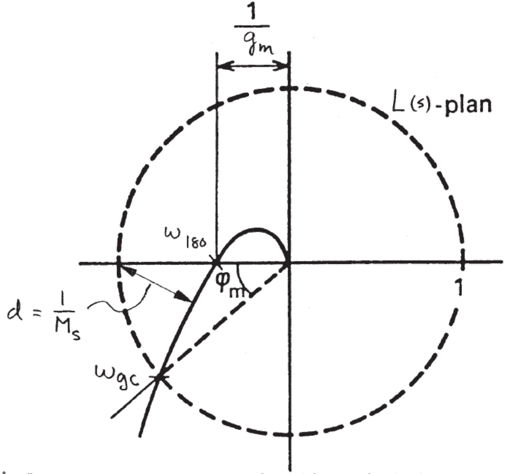
Performance specifications

Expression in the frequency-domain

- **Stability and robustness**: gain margin g_m, phase margin ϕ_m and stability margin s_m (maximum sensitivity $M_s = \frac{1}{s_m}$)



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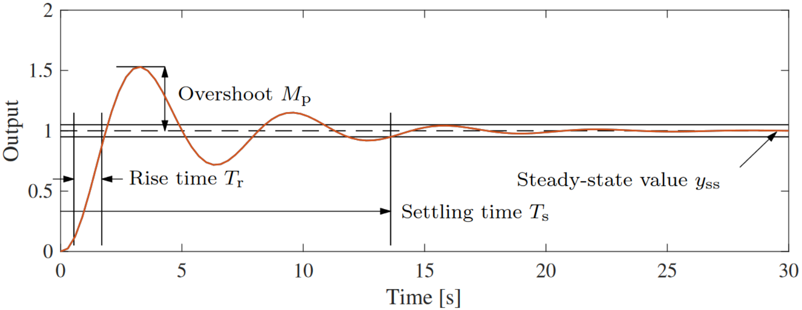
  - Time-domain: overshoot, rise time, settling time, steady state-error



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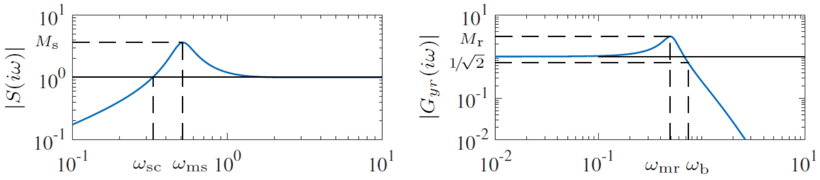

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$$P(s) = \frac{k}{\frac{1}{\omega_0^2}s^2 + \frac{2\xi}{\omega_0}s + 1}$$

$$\phi = arccos\xi$$

Property	Value
Steady-state value	k
Rise time	$T_r = e^{\phi/tan\phi}$
Overshoot	$M_p = e^{-\pi\xi\sqrt{1-\xi^2}}$
Settling time 2%	$T_s \sim \frac{1}{4\xi\omega_0}$

Performance specifications

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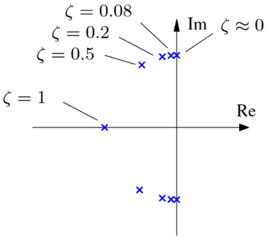
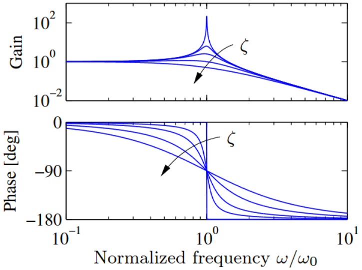


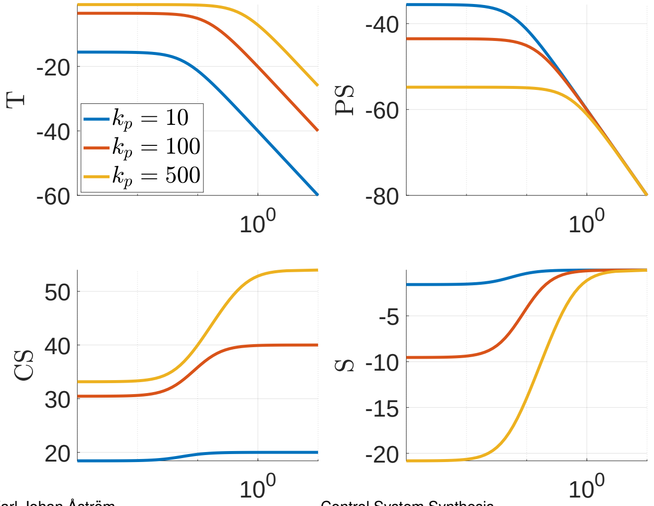
(a) Eigenvalues



(b) Frequency responses

Performance specifications

Expression in the frequency-domain



Performance specifications

Expression in the frequency-domain

$$C(s) = k_p + \frac{k_p}{s} \rightarrow T(s) = \frac{\frac{k_p}{k_i}s+1}{\frac{m}{k_i}s^2+\frac{b+k_p}{k_i}s+1}$$

- Overshoot $< 10\%$

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Performance specifications

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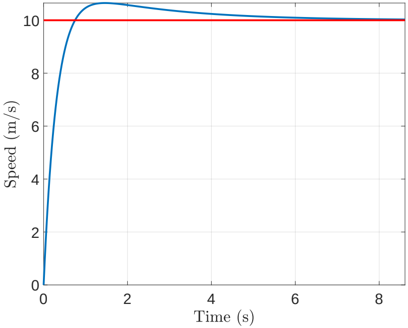

Figure: Reference tracking

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Performance specifications

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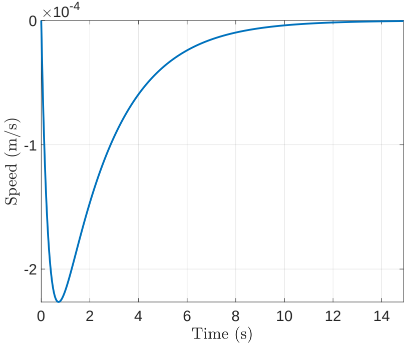
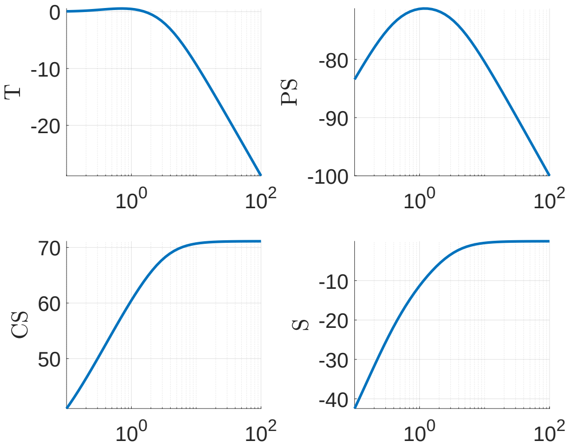


Figure: Disturbance rejection

Performance specifications

Expression in the frequency-domain



Performance specifications

Loopshaping

- Design technique in the frequency-domain
- Focus on the open-loop transfer function $L = PC \to C = L/P$ of high order

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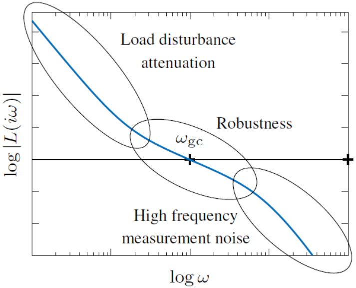
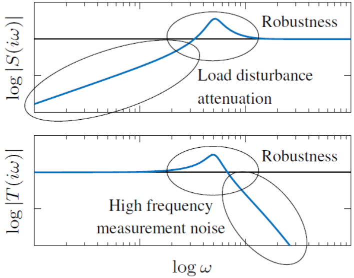


(a) Gain plot of loop transfer function



(b) Gain plot of sensitivity functions

1 Performance specifications

- Driving example: Cruise control
- Design trade-offs
- Expression in the frequency-domain
- Loopshaping

2 Fundamental limitations

- System design considerations
- Sensitivity minimization
- Bode's integral formula
- Gain crossover frequency inequality
- Internal stability

Fundamental limitations

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Fundamental limitations

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- unstable system $\rightarrow$ needs a fast controller (bandwidth of sensors and actuators)
- systems with time-delays $\rightarrow$ impossible to take fast control actions

$\rightarrow$ limitations often expressed by conditions on the poles and zeros of the system

- Can you rework on system design while doing control?

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Fundamental limitations
Sensitivity minimization

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Fundamental limitations

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- **Low static error:** $\min\limits_{S} \max\limits_{\omega \in (0,\omega_c)} |S(\jmath\omega)|$

Fundamental limitations

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- **Low static error:** $\min_S \max_{\omega \in (0,\omega_c)} \left|S(\jmath\omega)\right|$
  - ω_c is the cutoff frequency
  - $\lim_{t \to \infty} \epsilon(t) = \lim_{s \to 0} S(s)$

The sensitivity function cannot be made small over a wide frequency range.

- Bode's integral formula (invariant)

Fundamental limitations

Bode's integral formula

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Bode's integral formula

The sensitivity function cannot be made small over a wide frequency range.

- Bode's integral formula (invariant)

$\rightarrow$ reducing the sensitivity at one frequency increases it at another

- Right Half-Plane (RHP) poles in the process makes it worse
- Control design is always a compromise

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Fundamental limitations

Bode's integral formula

For an internally stable closed loop system, if $\lim_{s \to \infty} sL(s) = 0$:

$$\int_0^\infty log(|S(\jmath\omega)|)d\omega = \pi \sum p_k$$

where p_k are the RHP poles of the open-loop L.

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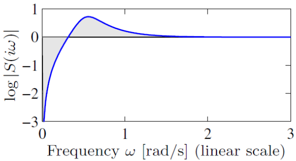

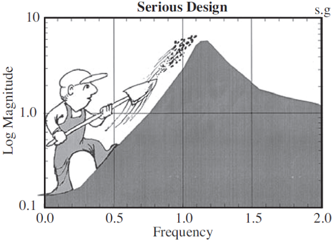

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Similarly:

$$\int_0^\infty \frac{log(|T(\jmath\omega)|)}{\omega^2} d\omega = \pi \sum \frac{1}{z_i}$$

where z_i are the RHP zeros of the open-loop L.

Fundamental limitations

Gain crossover frequency inequality

$$P(s) = P_{mp}(s)P_{nmp}(s)$$

$$|P_{mp}(\jmath\omega)| = 1$$

$$P(s) = \frac{s-2}{(s+1)(s-1)} = \frac{s+2}{(s+1)^2}\frac{(s-2)(s+1)}{(s+2)(s-1)} = P_{mp}P_{nmp}$$

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$$P(s) = P_{mp}(s)P_{nmp}(s)$$

$$|P_{mp}(\jmath\omega)| = 1$$

$$argL(\jmath\omega_{gc}) = argP_{nmp}(\jmath\omega_{gc}) + argP_{mp}(\jmath\omega_{gc}) + argC(\jmath\omega_{gc}) \geq -\pi + \phi_m$$

ϕ_m is the desired phase margin

Fundamental limitations

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ϕ_m is the desired phase margin

Assuming that C has no poles or zeros in the RHP:

$$argP_{mp}(\jmath\omega_{gc}) + argC(\jmath\omega_{gc}) = slope * \frac{\pi}{2}$$

Fundamental limitations

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ϕ_m is the desired phase margin

$$argP_{nmp}(\jmath\omega_{gc}) + slope * \frac{\pi}{2} \geq -\pi + \phi_m$$

$\rightarrow$ there is a trade-off between phase margin and speed

- Fast RHP poles $\rightarrow$ larger ω_{gc}
- Slow RHP zeros $\rightarrow$ lower ω_{gc}

Fundamental limitations

Internal stability

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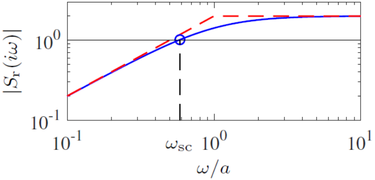
Maximum modulus principle: if G is bounded and analytic in the RHP:

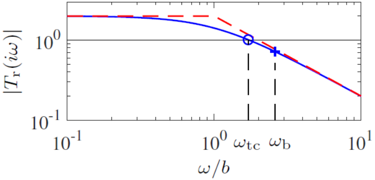
$$\max_{\omega \in \mathbb{R}} |G(j\omega)| = \max_{Re(s)\geq 0} |G(s)|$$

Fundamental limitations

Maximum modulus principle



(a) Requirements for sensitivity



(b) Requirements for complementary sensitivity

$$S_r(s) = \frac{M_s s}{s+a} \qquad T_r(s) = \frac{M_t b}{s+b}$$

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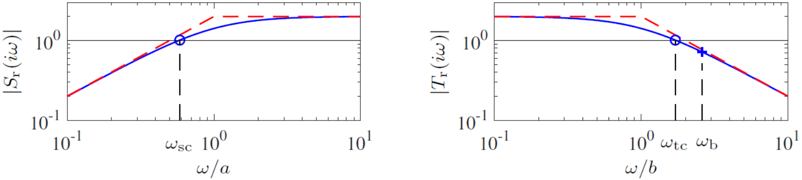

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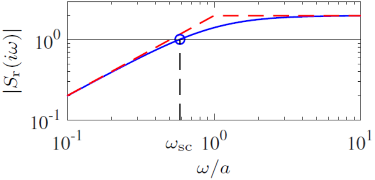

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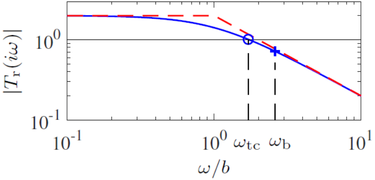

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$$a \leq z(M_s - 1) \rightarrow \omega_{sc} \leq z\sqrt{\frac{M_s-1}{M_s+1}}$$

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