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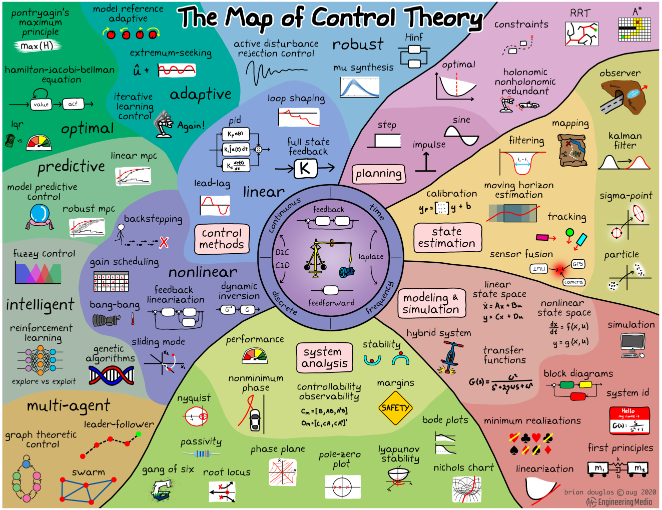
Control System Synthesis - Optimal control and LQG

PHD CLASS - FALL 2020


The Map of Control Theory
pontryagin's maximum principle
Max(H)
model reference adaptive
extremum-seeking
hamilton-jacobi-bellman equation
value
act
iterative learning control
Again!
adaptive
lqr
optimal
linear mpc
predictive
model predictive control
robust mpc
fuzzy control
intelligent
reinforcement learning
explore vs exploit
genetic algorithms
multi-agent
leader-follower
graph theoretic control
swarm
backstepping
gain scheduling
nonlinear
feedback linearization
dynamic inversion
bang-bang
sliding mode
active disturbance rejection control
robust
Hinf
mu synthesis
loop shaping
pid
full state feedback
lead-lag
linear
control methods
constraints
RRT
A*
optimal
holonomic
nonholonomic
redundant
observer
step
sine
impulse
planning
filtering
mapping
kalman filter
calibration
moving horizon estimation
state estimation
tracking
sigma-point
sensor fusion
IMU
GPS
camera
particle
continuous
time
feedback
D2C
C2D
laplace
feedforward
discrete
frequency
modeling & simulation
linear state space
nonlinear state space
simulation
hybrid system
transfer functions
block diagrams
system id
minimum realizations
first principles
linearization
performance
system analysis
stability
nonminimum phase
controllability observability
margins
SAFETY
nyquist
passivity
phase plane
pole-zero plot
lyapunov stability
bode plots
gang of six
root locus
nichols chart
brian douglas © aug 2020
Engineering Media

Content overview

The optimal control problem

$\rightarrow$ Find a control for a dynamical system over a period of time such that an objective function is optimized

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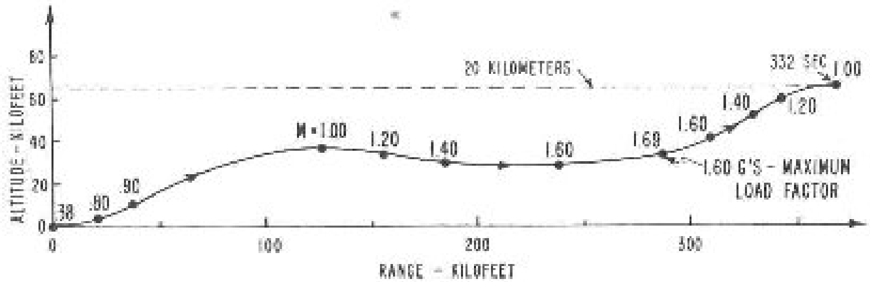
$\rightarrow$ Find a control for a dynamical system over a period of time such that an objective function is optimized

$$\underset{x(\cdot),u(\cdot)}{\text{minimize}} \int_0^T L(x(t),u(t))dt + E(x(T))$$

$$\text{subject to} \quad \begin{array}{ll} x(0) = x_0 & \text{(fixed initial value)} \\ \dot{x}(t) = f(x(t),u(t)) & \text{(model)} \\ h(x(t),u(t)) \geq 0 & \text{(path constraints)} \\ r(x(T)) \geq 0 & \text{(terminal constraints)} \end{array}$$

Brysons Flight Test

Bryson (Professor at Harvard) made major computations. A nice test was on an airplane



Twice as fast as with standard procedure!

The optimal control problem

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Motivations

- Get the best performances
- Design an acceptable control law (constraints enforcement)

The optimal control problem

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Ask yourself:

1. What is the optimization criteria?

The optimal control problem

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Motivations

- Get the best performances
- Design an acceptable control law (constraints enforcement)

Ask yourself:

1. What is the optimization criteria?
   - Different norms
   - Performance AND effort

The optimal control problem

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Motivations

- Get the best performances
- Design an acceptable control law (constraints enforcement)

Ask yourself:

1. What is the optimization criteria?
2. What are the optimization variables?
   - controllers
   - control signals

The optimal control problem

- application of optimization to control
- work of Lev Pontryagin and Richard Bellman in the 1950s

The optimal control problem

- application of optimization to control
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Three types of resolution:

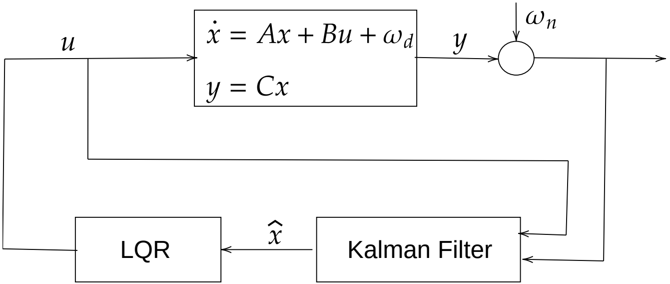
- Hamilton-Jacobi-Bellman (HJB) and dynamic programming
- Indirect methods and Pontryagin's principle
- Direct methods

$\rightarrow$ second part of the lecture

LQG control

What is LQG control?

Linear **Q**uadratic **G**aussian control



LQG control
Controllability and LQR

$$\dot{x} = Ax + Bu$$
$$u = -K_r x$$

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$$\dot{x} = (A - BK_r)\, x$$

LQG control

Controllability and LQR

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- possibility to place the eigenvalues of the controlled system

$\rightarrow$ basic idea of pole placement

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- If the system is **controllable**, eigenvalues can be placed anywhere

LQG control

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$\rightarrow$ basic idea of pole placement

- If the system is **controllable**, eigenvalues can be placed anywhere

$$\mathcal{C} = \begin{bmatrix} B \; AB \; \dots \; A^{n-1}B \end{bmatrix}$$
$$\mathrm{rank}(\mathcal{C}) = n$$

How to design K_r?

- pole placement
- LQR

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$$J(t) = \int_0^T x(t)Qx(t) + u(t)Ru(t)dt$$

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$$J(t) = \int_0^T x(t)Qx(t) + u(t)Ru(t)dt$$

Analytical solution: $K_r = R^{-1}B^\star X$

$$A^\star X + XA - XBR^{-1}B^\star X + Q = 0$$

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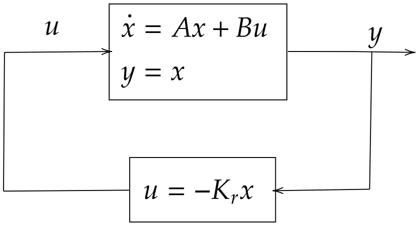
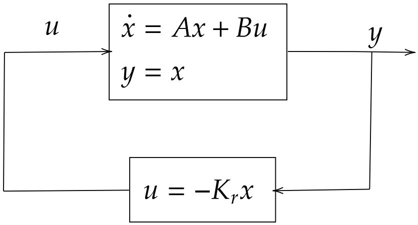
- Solving this Riccati equation is costly ($\mathcal{O}(n^3)$)
- In Matlab: $K_r = lqr(A, B, Q, R)$;

LQG control



- State is not always accessible

LQG control



- State is not always accessible

$\rightarrow$ Need for an estimator: Kalman Filter

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System:

$$\begin{cases} \dot{x} = Ax + Bu + \omega_d \\ y = Cx + \omega_d \end{cases}$$

with gaussian noise ω_n and disturbance ω_d with zero mean and known covariances V_n and V_d

LQG control

Observability and state estimation

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$$\begin{cases} \dot{x} = Ax + Bu + \omega_d \\ y = Cx + \omega_d \end{cases}$$

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The system is **observable** if t is possible to estimate any state $x \in \mathbb{R}^n$ from a time-history of the measurements $y(t)$

$$\mathcal{O} = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}$$

$$\text{rank}\mathcal{O} = n$$

LQG control

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Estimator:
$$\begin{cases} \dot{\hat{x}} = A\hat{x} + Bu + K_f(y - \hat{y}) \\ \hat{y} = C\hat{x} \end{cases}$$

LQG control

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Design of K_f to minimize $\lim_{t \to \infty} \mathbb{E}((x(t) - \hat{x}(t))^\star (x(t) - \hat{x}(t)))$

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$$K_f = YC^\star V_n$$

$$YA^\star + AY - YC^\star V_n^{-1} CY + V_d = 0$$

LQG control

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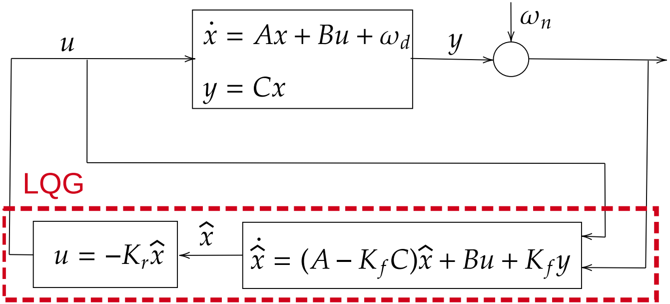
- In Matlab: $K_f = lqe(A, V_d, C, V_d, V_n);$
- Link with LQR: $K_f = (lqr(A', C', V_d, V_n))';$

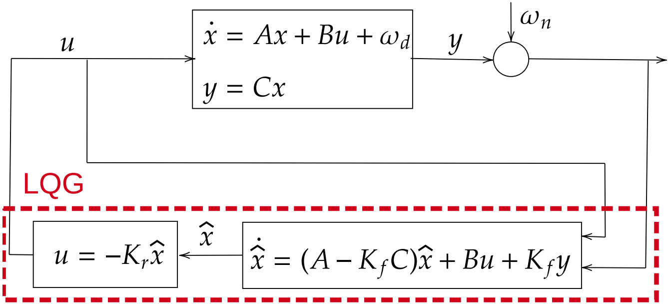
LQG control

Summary



LQG control

Summary



$$\begin{pmatrix} \dot{x} \\ \dot{\varepsilon} \end{pmatrix} = \begin{pmatrix} A - BK_r & BK_r \\ 0 & A - K_f C \end{pmatrix} \begin{pmatrix} x \\ \varepsilon \end{pmatrix} + \begin{pmatrix} I & 0 \\ I & -K_f \end{pmatrix} \begin{pmatrix} \omega_d \\ \omega_n \end{pmatrix}$$

$$\varepsilon = x - \hat{x}$$

LQG control

Summary

- The closed-loop eigenvalues of the LQG regulated system are optimally chosen through K_r and K_f

LQG control

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- Handles MIMO, multi timescales and optimality

LQG control

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- The closed-loop eigenvalues of the LQG regulated system are optimally chosen through K_r and K_f
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- Handles MIMO, multi timescales and optimality
- Lack of robustness

Guaranteed Margins for LQG Regulators

JOHN C. DOYLE

Abstract—There are none.

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LQG control

Summary

- The closed-loop eigenvalues of the LQG regulated system are optimally chosen through K_r and K_f
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  - Loop transfer recovery

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LQG control

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- The closed-loop eigenvalues of the LQG regulated system are optimally chosen through K_r and K_f
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- Handles MIMO, multi timescales and optimality
- Lack of robustness
  - Loop transfer recovery
  - **Motivation for the development of robust control**

Optimal control

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$$\underset{x(\cdot),u(\cdot)}{\text{minimize}} \int_0^T L(x(t),u(t))dt + E(x(T))$$

$$\begin{array}{lll} \text{subject to} & x(0) = x_0 & \text{(fixed initial value)} \\ & \dot{x}(t) = f(x(t),u(t)) & \text{(model)} \\ & h(x(t),u(t)) \geq 0 & \text{(path constraints)} \\ & r(x(T)) \geq 0 & \text{(terminal constraints)} \end{array}$$

1. Cost-to-go function
$J(\overline{x}, \overline{t}) = \min_{x,u} \int_{\overline{t}}^{T} L(x,u)dt + E(x(T))$ s.t. $x(\overline{t}) = \overline{x}, \ldots$

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2. Principle of optimality (**sufficient condition**) on grid $t_0 = 0 < \cdots < t_N = T$
$$J(x_k, t_k) = \min_{x,u} \int_{t_k}^{t_{k+1}} L(x,u)dt + J(x_{k+1}, t_{k+1}) \text{ s.t. } x(t_k) = x_k, \ldots$$

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1. Cost-to-go function
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3. Dynamic programming recursion
   1. Start with $J(x, t_N) = E(x)$
   2. Compute $J(x_k, t_k)$ recursively backwards, for $k = N-1, \ldots, 0$

Optimal control

Dynamic programming and HJB

- For infinitely small time steps, we obtain the HJB equation:

$$-\frac{\partial J}{\partial t}(x,t) = \min_{u}\left(L(x,u) + \frac{\partial J}{\partial x}(x,t)f(x,u)\right) \text{ subject to } h(x,u) \geq 0$$

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- Backward resolution of this PDE starting with $J(x,T) = E(x)$

Optimal control

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- Backward resolution of this PDE starting with $J(x,T) = E(x)$
- At time t, the optimal control decision is given by:

$$u^\star(x,t) = \arg\min_u \left(L(x,u) + \frac{\partial J}{\partial x}(x,t)f(x,u)\right) \text{ s.t. } h(x,u) \geq 0.$$

Optimal control

Indirect methods and Pontryagin's principle

$$u^\star(x,t) = \arg\min_u \left(L(x,u) + \frac{\partial J}{\partial x}(x,t) f(x,u) \right) \text{ s.t. } h(x,u) \geq 0$$

- Introduce an adjoint variable $\lambda(t) = \frac{\partial J}{\partial x}(x,t)^T$

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Optimal control

Indirect methods and Pontryagin's principle

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- Definition of the Hamiltonian: $H(x,u,\lambda) = L(x,u) + \lambda^T f(x,u)$

Optimal control

Indirect methods and Pontryagin's principle

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- Pontryargin's Maximum Principle (necessary condition):

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Optimal control

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- Adjoint equation (differentiation of HJB equation)

$$-\dot{\lambda}^T = \frac{\partial}{\partial x}(H(x(t), u^\star(t,x,\lambda), \lambda(t)))$$

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Optimal control

Indirect methods and Pontryagin's principle

$$u^\star(x,t) = \arg\min_u \left(L(x,u) + \frac{\partial J}{\partial x}(x,t) f(x,u) \right) \text{ s.t. } h(x,u) \geq 0$$

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- Adjoint equation (differentiation of HJB equation)

$$-\dot{\lambda}^T = \frac{\partial}{\partial x}(H(x(t),u^\star(t,x,\lambda),\lambda(t)))$$

- Reformulation of the problem that can be solved using gradient techniques, shooting methods, ...

Optimal control

Summary

$$\frac{dx}{dt} = f(x,u), \qquad \min_u J(u) = G(x(T)) + \int_0^T g(x(t),u(t))dt$$

Hamiltonian

$$H(x,p,u) = g(x,u) + p^T f(x,u), \qquad H^0(x,p) = \min_u H(x,p,u)$$

Euler-Lagrange-Pontryagin (particle view)

$$\frac{dx}{dt} = \frac{\partial H^0}{\partial p}, \qquad x(0) = a$$

$$\frac{dp}{dt} = -\frac{\partial H^0}{\partial p}, \qquad p(T) = G'(x)$$

Hamilton-Jacobi-Bellman (wave view)

$$\frac{\partial V}{\partial t} + H^0\left(x, \frac{\partial V}{\partial x}\right), \qquad V(x,T) = G(x)$$

Optimal control

Summary

- Small-scale systems
  - Dynamic Programming in the discrete case
  - HJB in the continuous case

  $\rightarrow$ search over the whole state space, find global optimum and optimal control is precomputed
- Large scale system
  - Indirect methods
    - Reformulation in a boundary value problem with only $2n$ ODEs
    - Only necessary conditions for local optimality and need an explicit expression for $u^\star$
  - Direct methods: discretize then optimize
    - Transformation into finite dimensional Non Linear Programming Problem (NLP)
    - Can use state-of-the-art methods for NLP solution.
    - Easier way to handle the constraints
    - Obtains only suboptimal/approximate solution.

Optimal Control at the Department

- Krister Mårtensson (TD #2 1972)
- Torkel Glad (TD #11, 1976)
- Bengt Pettersson (TL #2, 1970)
- Bo Lincoln (TL #67, 2003)
- Sven Hedlund (TL #68 2003)
- Mattias Grundelius (TD #71 1995)
- Johan Åkesson (TD #81 2007)
- Andreas Wernerud (TD #82 2008)
- Per-Ola Larsson (TD #88 2011)
- Karl Mårtensson (TD #91 2012)
- Pontus Gisselson (TD #94 2012)