Lecture 6 – Nonlinear controllability

Nonlinear Controllability

Material

Lecture slides

Handout from Nonlinear Control Theory, Torkel Glad (Linköping)

Handout about Inverse function theorem by Hörmander

Nonlinear System

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x,u) \end{aligned}$$

Important special affine case:

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned}$$

f : drift term

g : input term(s)

What you will learn today (spoiler alert)

New mathematical concepts and language

$$\text{Manifolds, charts } (M, \phi(x))$$

$$\text{Vector fields } \sum_{i=1}^{n} a_i(x) \frac{\partial}{\partial x_i}$$

$$\text{Lie-derivative } L_X(f) = \sum_{i=1}^{n} a_i(x) \frac{\partial f}{\partial x_i}$$

$$\text{Lie-bracket } [f,g] = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g$$

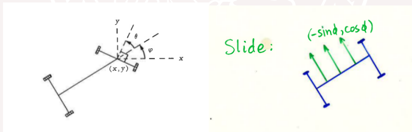
What you will learn today

Local Controllability:

- *A nonlinear system is controllable if the linearized system is controllable.*
- $\dot{x} = f(x) + g(x)u$ *is "accessible" iff*

$$\text{dim } (f, g, [f,g], [f,[f,g]], \ldots) = n$$

Fundamental Parking Theorem



What we will not do

Local Observability. Depends on x_0 and u.

$y_j = h_j(x)$

$\mathcal{O} = \text{span} L_{X_1} \ldots L_{X_k} h_j(x)$

$d\mathcal{O} = \text{span } (dH \mid H \in \mathcal{O})$

The system is locally observable if

$$\dim\, (d\mathcal{O}) = n$$

Duality between observability and controllability

And not this

Nonlinear Kalman Decomposition

Can find coordinates (x_1, x_2, x_3, x_4) so that

$$\begin{aligned} \dot{x}_1 &= f^1(x_1, x_3) + g(x_1, x_3)u \\ \dot{x}_2 &= f^2(x_1, x_2, x_3, x_4) + g(x_1, x_2, x_3, x_4)u \\ \dot{x}_3 &= f^3(x_3) \\ \dot{x}_4 &= f^4(x_1, x_3) \\ y &= h(x_1, x_3) \end{aligned}$$

Relative Degree Smallest r such that $L_g L_f^{r-1} h(x_0) \neq 0$

Exact Linearization by Feedback

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ u &= \alpha(x) + \beta(x)v \text{ and } z = Z(x) \implies \dot{z} = Az + Bv \end{aligned}$$

System is feedback linearizable if one can find $y = h(x)$ *so the system has relative degree* n. *Can be checked with Lie-brackets*

and not this

Differential Flatness

Zero Dynamics

Nonlinear Minimum Phase

Disturbance Decoupling

Normal Forms

Stabilization

Basic Result: Linearization at (x_0, u_0)

$$\dot{x} = f(x) + g(x)u, \qquad x(0) = x_0$$

Theorem Suppose $f(x_0) + g(x_0)u_0 = 0$. If the linearization

$$\begin{aligned} \dot{z} &= Az + Bv \\ A &= \frac{\partial f}{\partial x}(x_0) + \frac{\partial g}{\partial x}(x_0)u_0 \\ B &= g(x_0) \end{aligned}$$

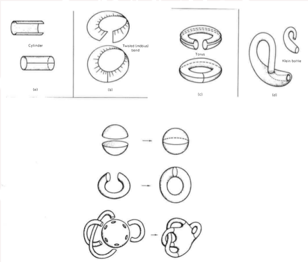
is controllable, then for all $T > 0, \epsilon > 0$ the set

$$X_{T,\epsilon} = \{x(T);\ |u - u_0| < \epsilon\}$$

contains a neighborhood of x_0. (Proof: Nice exercise in using the inverse function theorem)

Manifolds

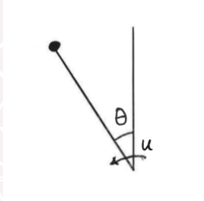
What are natural mathematical models for state spaces?



Piece together "bent" pieces of R^n.

Same local properties as R^n. Different globally

Example - Pendulum

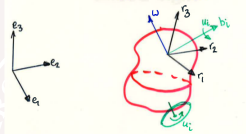


$$\ddot{\theta} = \sin(\theta) + u$$

Natural state space: $R \times S^1$ = cylinder

$S^1 =$ unit circle

Rigid Bodies

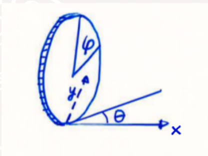


Natural State Space

$$\begin{aligned} R &= \begin{pmatrix} r_1 & r_2 & r_3 \end{pmatrix} \in SO(3) \\ \text{ie } RR^T &= I \text{ and } \det(R) = 1 \\ \dot{R} &= -R \times w \Leftrightarrow \dot{R} = -RS(w) \\ S(w) &= \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix} \end{aligned}$$

Rolling Penny

$$\begin{aligned} \dot{x} &= u_1 \cos(\theta) \\ \dot{y} &= u_1 \sin(\theta) \\ \dot{\varphi} &= u_1 \\ \dot{\theta} &= u_2 \end{aligned}$$



The linearization is not controllable (check)

Can the penny be moved sideways in small time (keeping the head up)?

Rolling Penny

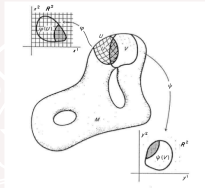
Yes it can. But it is not obvious.

The penny has non-holonomic constraints $a(z)\dot{z} = 0$

$$\begin{pmatrix} \sin\theta & -\cos\theta & 0 & 0 \\ \cos\theta & \sin\theta & -1 & 0 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\varphi} \\ \dot{\theta} \end{pmatrix} = 0$$

Can not be written as holonomic constraints: $h(z) = 0 \Longrightarrow h_z \dot{z} = 0$.

Definition of Manifold



A C^∞ (=smooth) manifold is a topological space M together with an atlas $\{U_\alpha, \varphi_\alpha\}$ of pairwise C^∞-compatible coordinate charts that cover M.

Topological space M?

Atlas $\{U_\alpha, \varphi_\alpha\}$?

Pairwise C^∞-compatible coordinate charts ?

Topology

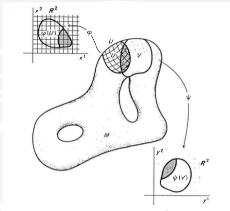
A topology on a set M is a collection T of subsets of M.

O is called "open" if $O \in T$.

The collection T must be such that

- $\emptyset, M \in T$
- $O_1, O_2 \in T \Longrightarrow O_1 \cap O_2 \in T$
- $\{O_i\} \in T \Longrightarrow \cup O_i \in T$

Compatible Coordinate Charts



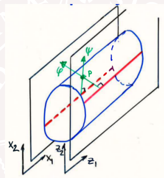
Compatible: $\psi \circ \varphi^{-1}(x) \in C^\infty$

f is called "smooth" if $f(\psi^{-1}(x)) \in C^\infty, \quad \forall \psi$

Note: $f \circ \varphi^{-1}(x) = f \circ \psi^{-1} \circ \psi \circ \varphi^{-1} \in C^\infty$

Independent on coordinate charts.

Example: Cylinder

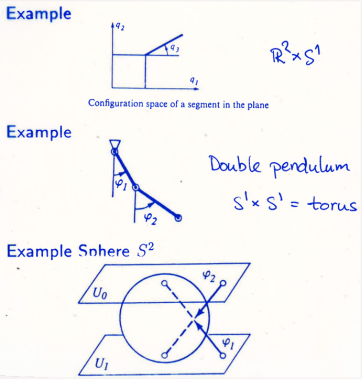


$\psi \circ \varphi^{-1}$ smooth on $U \cap V = (x_2 \neq 0, z_2 \neq 0)$

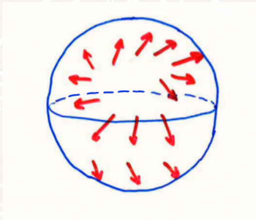
$z = \psi(\varphi^{-1}(x))$ is given by $(z_1, z_2) = (x_1, 4/x_2)$

The cylinder is a smooth manifold

Examples



Global Differences to R^n - Example



Any smooth velocity field v on S^2 must have a point where $v(x) = 0$

"You can't comb the hair of a tennis ball"

Manifolds defined by equation systems

Many manifolds are defined implicitly by equations systems

$$\begin{aligned} f_1(x_1, \dots, x_n) &= 0 \\ &\vdots \\ f_k(x_1, \dots, x_n) &= 0 \end{aligned}$$

When does this describe a (smooth) $n-k$-dimensional manifold?

Differentials

$f : A \to B$ is called differentiable at $x \in A$ iff there is a continuous linear map $DF_x(h) : A \to B$ such that

$$\|f(x+h) - f(x) - DF_x(h)\| \to 0, \quad h \to 0$$

DF_x = differential (Jacobian)

$$DF = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots \\ \frac{\partial f_2}{\partial x_1} & \ddots & \\ \vdots & & \end{pmatrix}$$

Differentials

$$DF = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots \\ \frac{\partial f_2}{\partial x_1} & \ddots & \\ \vdots & & \end{pmatrix}$$

Definition Rank of f at x := rank(DF_x).

If f smooth then

$$\text{Rank } (DF_{x_0}) = k \implies \text{Rank } (DF_x) \geq k$$

for all x close to x_0.

Proof: $D_k(x) = k \times k$ submatrix of DF_x with $\det(D_k(x_0)) \neq 0 \implies \det(D_k(x)) \neq 0$, for x close to x_0.

Inverse Function Theorem

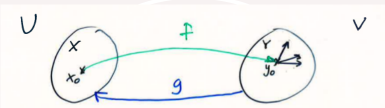
Theorem Let X be open in U and $f \in C^1(X, V)$, $f(x_0) = y_0$. For existence of $g \in C^1(Y, U)$ where Y is a neighborhood of y_0 so

a) $f \circ g =$ identity near y_0
b) $g \circ f =$ identity near x_0
c) a) and b)

it is necessary and sufficient that there is a linear map A such that respectively

a') $f'(x_0)A = I_V$
b') $Af'(x_0) = I_U$
c') a') and b')

Condition c' implies that g is uniquely determined near y_0.



Proof idea: To solve $y = f(x)$ use

$$x_k = x_{k-1} + f'(x_0)^{-1}(y - f(x_{k-1}))$$

Prove $\sum(x_k - x_{k-1})$ converges for y near y_0.

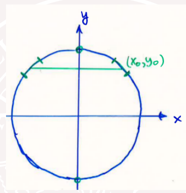
See handout.

Implicit Function Theorem

$$h(x, y) = 0$$

$$\frac{\partial h}{\partial x} \text{ full rank} \implies x = x(y) \text{ uniquely}$$

Example



$$h(x, y) = x^2 + y^2 - 1, \quad h'_x = 2x$$

So $x = x(y)$ uniquely except near $(0, \pm 1)$.

In fact $x = \sqrt{1 - y^2}, \ x_0 > 0$ and $x = -\sqrt{1 - y^2}, \ x_0 < 0$.

Discussion

Implicit F. T. $\implies$ Inverse F. T. c).

$$h(x,y) = y - f(x); \quad h'_x = f'_x \implies x = x(y) \text{ uniquely}$$

Inverse F. T. c) $\implies$ Implicit F. T.

$$f(x,y) = (h(x,y), y)$$

$$f' = \begin{pmatrix} h'_x & h'_y \\ 0 & I \end{pmatrix} \quad \text{full rank}$$

So (x,y) locally determined by $(h,y) = (0,y)$

$$\implies x = x(y) \text{ uniquely locally}$$

Note

$$f(x,y) = 0$$

$$f'_x \frac{\partial x}{\partial y} + f'_y = 0$$

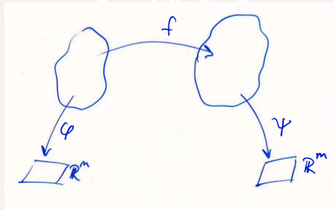
$$\frac{\partial x}{\partial y} = -(f'_x)^{-1} f'_y$$

Try it yourself example

$$\begin{aligned} x_1^3 - e^{x_2} + x_3^3 - 1 &= 0 \\ x_1^2 + x_2 - x_3^2 &= 0 \end{aligned}$$

Are x_1, x_2 smooth functions of x_3 around $(1,0,1)$?
What is $\frac{\partial x_1}{\partial x_3}$ at that point?

Functions Between Manifolds



Definition

$$f \in C^\infty \iff \psi \circ f \circ \varphi^{-1} \in C^\infty, \quad \forall \psi, \varphi$$

Manifolds by equation systems

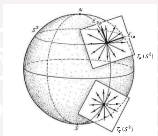
$$\begin{aligned} f_1(x_1, \ldots, x_n) &= 0 \\ &\vdots \\ f_k(x_1, \ldots, x_n) &= 0 \end{aligned}$$

determines an $n-k$ dimensional manifold near $\bar{x}$ if

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots \\ \frac{\partial f_2}{\partial x_1} & \ddots & \\ \vdots & & \end{pmatrix} \quad \text{has full rank } (=k) \text{ at } \bar{x}$$

("Gradients $\nabla f_1, \ldots, \nabla f_k$ are linearly independent")

Tangent Vectors - different definitions



- Define it only for manifolds embedded in R^n:

$$\dot{x} = \lim_{t \to 0} \frac{\varphi(t) - \varphi(0)}{t}$$

  Velocity vectors in R^n.
- Coordinate free version. Tangent vectors at $x \leftrightarrow$ "equivalance classes of curves with $\varphi(0) = x$", we define $\varphi(t) \sim \psi(t)$ when

$$\varphi(0) = \psi(0) = x \quad \text{and} \quad \lim_{t \to 0} \frac{\varphi(t) - \psi(t)}{t} = 0 \quad \text{in some chart}$$

Our Definition

Derivative operator $X(f) : (f : M \mapsto R) \mapsto R$

$$\begin{aligned} X(\alpha f + \beta g) &= \alpha X(f) + \beta X(g) \\ X(fg) &= f X(g) + g X(f) \end{aligned}$$

Example: Take any coordinate chart (U, φ) with coordinates x. Then

$$X_a = \sum_{i=1}^{n} \alpha_i \frac{\partial}{\partial x_i}$$

is a tangent vector, where

$$X_a(f) = \sum_{i=1}^{n} \alpha_i \frac{\partial f}{\partial x_i}(a)$$

Different notation

$$L_X(f) = X(f) \quad \text{Lie-derivative = fishermans derivative}$$

Examples

$$\frac{\partial}{\partial \theta}; \quad \frac{\partial}{\partial z}; \quad z\frac{\partial}{\partial \theta} + \sin(\theta)\frac{\partial}{\partial z}$$

Coordinate Change

$$X = \begin{pmatrix} \frac{\partial}{\partial x_1} & \cdots & \frac{\partial}{\partial x_n} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

Change coordinates $\beta = \frac{\partial z}{\partial x} \alpha$ or

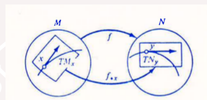
$$\begin{pmatrix} \frac{\partial}{\partial x_1} & \cdots & \frac{\partial}{\partial x_n} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial z_1} & \cdots & \frac{\partial}{\partial z_n} \end{pmatrix} \frac{\partial z}{\partial x}$$

Example

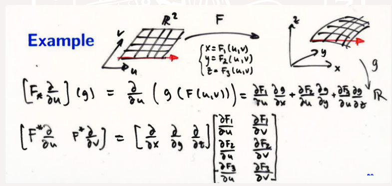
$$\begin{aligned} z_1 &= x_1 \\ z_2 &= x_1 + x_2 \\ \frac{\partial}{\partial x_1} &= \frac{\partial z_1}{\partial x_1}\frac{\partial}{\partial z_1} + \frac{\partial z_2}{\partial x_1}\frac{\partial}{\partial z_2} = \frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} \end{aligned}$$

Note that $x_1 = z_1$ does not imply $\frac{\partial}{\partial x_1} = \frac{\partial}{\partial z_1}$

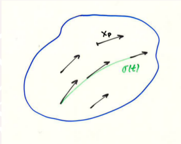
Push Forward Operator



$$[f_* X](g) := X(g \circ f)$$



(Smooth) Vector Fields



Assigns a tangent vector to each point: $p \mapsto X_p$

$$X = \sum_{i=1}^{n} X_i(p) \frac{\partial}{\partial x_i}$$

$X_i(p)$ smooth functions of p.

$$\text{Alternativ notation:} \quad X \sim \begin{pmatrix} X_1(x_1, \dots, x_n) \\ \vdots \\ X_n(x_1, \dots, x_n) \end{pmatrix}$$

Integral Curve

$\sigma(t)$ is an *integral curve* to X if in local coordinates

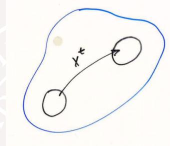
$$\begin{aligned} \sigma(t) &= \begin{pmatrix} \sigma_1(t) \\ \vdots \\ \sigma_n(t) \end{pmatrix} \\ \frac{\partial}{\partial t}(g(\sigma(t))) &= X(\sigma(t))(g) \\ \sum \frac{\partial g}{\partial x_i} \frac{d\sigma_i}{dt} &= \sum X_i(\sigma(t)) \frac{\partial g}{\partial x_i} \end{aligned}$$

i.e.

$$\begin{aligned} \dot\sigma_1 &= X_1(\sigma(t)) \\ &\vdots \\ \dot\sigma_n &= X_n(\sigma(t)) \end{aligned}$$

A set of ODEs

Transformation Group, Flow



$X^t(p) =$ solution to $\dot x = X(x), x(0) = p$

X^t is smooth. $X^0 =$ id

$$L_X(g) = X(g) = \sum_{i=1}^{n} X_i \frac{\partial g}{\partial x_i} = \lim_{h \to 0} \frac{g(X^h(p)) - g(p)}{h}$$

$L_{\alpha X + \beta Y} = \alpha L_X + \beta L_Y, \ \ \alpha, \beta \in R$

$$\dot x = f(x, u) \qquad f : M \times U \mapsto TM$$

Example

$$\begin{aligned} \dot x &= f(x) + g(x)u \\ y &= h(x) \end{aligned}$$

$$\begin{aligned} \dot y &= \frac{\partial h}{\partial x}\dot x = \frac{\partial h}{\partial x}(f + gu) = L_{f+gu}h \\ &= L_f h + u L_g h \\ y^{(k)} &= (L_{f+gu})^k h \end{aligned}$$

Main new object: Lie Bracket of vector fields

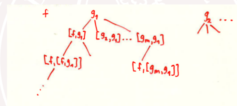
Consider two vector fields $\dot x = f(x)$ and $\dot x = g(x)$

Lie-bracket. New vector field

$$[f, g] = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g$$

Why is it interesting?

$$\dot x = f(x) + g_1(x)u_1 + g_2(x)u_2 + \dots$$



Roughly we have:

If the Liebracket "tree" has full rank, then the system is "controllable".

Example

$$\begin{aligned} \dot x_1 &= u_1 \\ \dot x_2 &= u_2 \\ \dot x_3 &= x_1 u_2 \pm x_2 u_1 \end{aligned}$$

This means $g_1 = \begin{pmatrix} 1 \\ 0 \\ \pm x_2 \end{pmatrix}$ and $g_2 = \begin{pmatrix} 0 \\ 1 \\ x_1 \end{pmatrix}$

$$[g_1, g_2] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \pm x_2 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \pm 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ x_1 \end{pmatrix}$$

Example

Hence at $x = 0$ we have

$$g_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad [g_1, g_2] = \begin{pmatrix} 0 \\ 0 \\ 1 - \pm 1 \end{pmatrix}$$

With the minus-sign the three vector fields span R^3, and we have controllability.

With the plus-sign the system is not controllable, in fact it can be seen that $x_1^2 + x_2^2 - 2x_3$ is an invariant.

Example

$$\begin{aligned} \dot x &= f(x) + g(x)u \\ y &= h(x) \end{aligned}$$

$$\begin{aligned} \dot y &= \frac{\partial h}{\partial x}\dot x = \frac{\partial h}{\partial x}(f + gu) = L_{f+gu}h \\ &= L_f h + u L_g h \\ y^{(k)} &= (L_{f+gu})^k h \end{aligned}$$

Lie-Brackets

$$[X,Y]_p(f) = X_p(Y(f)) - Y_p(X(f))$$

$$X \sim \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}; \qquad Y \sim \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix}$$

$$[X,Y] = \frac{\partial Y}{\partial x} X - \frac{\partial X}{\partial x} Y$$

Another example

$$\begin{aligned} X &= \begin{pmatrix} \cos\phi \\ r \end{pmatrix} \sim \cos\phi \frac{\partial}{\partial r} + r \frac{\partial}{\partial \phi} \\ Y &= \begin{pmatrix} r \\ 1 \end{pmatrix} \sim r\frac{\partial}{\partial r} + \frac{\partial}{\partial \phi} \\ [X,Y] &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\phi \\ r \end{pmatrix} - \begin{pmatrix} 0 & -\sin\phi \\ 1 & 0 \end{pmatrix} \begin{pmatrix} r \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos\phi - \sin\phi \\ -r \end{pmatrix} \sim (\cos\phi - \sin\phi)\frac{\partial}{\partial r} - r\frac{\partial}{\partial \phi} \end{aligned}$$

Lie-Brackets

Why are Lie-brackets so fundamental?

$\dot{x} = g_1 u_1 + g_2 u_2$

$$\begin{aligned} (u_1(t), u_2(t)) &= \begin{cases} (1,0) & t \in [0,h) \\ (0,1) & t \in [h, 2h) \\ (-1,0) & t \in [2h, 3h) \\ (0,-1) & t \in [3h, 4h) \end{cases} \\ x(4h) &= x_0 + h^2[g_1, g_2] + O(h^3) \end{aligned}$$

Trotters Product Formula

$$\Phi^t_{[X,Y]} = \lim_{n\to\infty} \left(\Phi^{\sqrt{\frac{t}{n}}}_{-Y} \Phi^{\sqrt{\frac{t}{n}}}_{-X} \Phi^{\sqrt{\frac{t}{n}}}_{Y} \Phi^{\sqrt{\frac{t}{n}}}_{X} \right)^n$$

Proof sketch

$$\left(1 + \frac{tf}{n} + o\left(\frac{tf}{n}\right)\right)^n \to e^{tf}$$

Some Lie-Bracket Formulas

$$[fX, gY] = fg[X,Y] + fX(g)Y - gY(f)X$$
$$[X,Y] = -[Y,X]$$
$$[X_1,[X_2,X_3]] + [X_2,[X_3,X_1]] + [X_3,[X_1,X_2]] = 0$$
$$L_X Y = [X,Y] = \lim_{h\to 0} \frac{1}{h}[X_*^{-h} Y - Y]$$

$$X_*^{-h} Y = \sum_{n=0}^{\infty} \mathrm{ad}_X^n Y \frac{h^n}{n!} = Y + h[X,Y] + \frac{h^2}{2}[X,[X,Y]] \dots$$

related to

$$e^A e^B = e^C; \qquad C = A + B + \frac{1}{2}[A,B] + \dots$$

Vector Fields, Summary

A vector field X is associated with

a) A system of differential equations
$$\frac{dx}{dt} = X(x)$$
b) A flow $\Phi^t : M \mapsto M, t \in [t_0, t_1]$, where $\sigma(t) = \Phi^t(x)$ is the solution to
$$\frac{d\sigma}{dt} = X(\sigma), \qquad \sigma(0) = x$$
c) A directional derivative
$$X_x f = \frac{d}{dt} f(\Phi^t(x))\Big|_{t=0}$$
d) A "derivation" of the algebra $C^\infty(M)$.
e) A partial differential operator
$$X = \sum X_j \frac{\partial}{\partial x_j}$$

$a \to b$ solution to differential equations

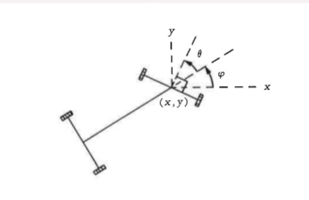
$b \to c$ direct

$c \to d$ direct

$d \to e$ proposition

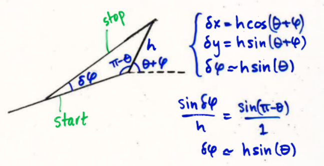
$e \to a$ direct

Park Your Car Using Lie-Brackets!



(x,y)	:	position
ϕ	:	direction of car
θ	:	direction of wheels
(x,y,ϕ,θ)	$\in$	$R^2 \times S^1 \times [\theta_{\min}, \theta_{\max}]$

Parking cont'd

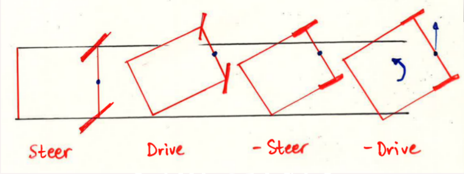


$$\text{Steer} := \frac{\partial}{\partial \theta}$$

$$\text{Drive} := \cos(\phi+\theta)\frac{\partial}{\partial x} + \sin(\phi+\theta)\frac{\partial}{\partial y} + \sin(\theta)\frac{\partial}{\partial \phi}$$

$$\text{Steer} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \text{Drive} = \begin{pmatrix} \cos(\phi+\theta) \\ \sin(\phi+\theta) \\ \sin(\theta) \\ 0 \end{pmatrix}$$

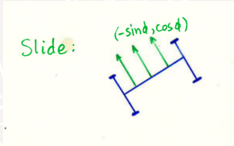
$$[\text{Steer}, \text{Drive}] = \dots = \begin{pmatrix} -\sin(\phi+\theta) \\ \cos(\phi+\theta) \\ \cos(\theta) \\ 0 \end{pmatrix} := \text{Wriggle}$$



An easy calculation (exercise) shows that

$$[\text{Wriggle, Drive}] = \begin{pmatrix} -\sin(\phi+\theta) \\ \cos(\phi+\theta) \\ 0 \\ 0 \end{pmatrix} =: \text{Slide}$$

For $\theta = 0$ this takes you sideways:



$$\text{Slide}^t(x, y, \phi, 0) = (x - t\sin(\phi), x + t\cos(\phi), \phi, 0)$$

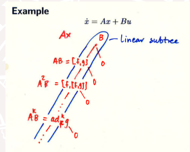
Fundamental Parking Theorem

You can get out of any parking lot that is larger than the car. Use the following control: Wriggle, Drive, –Wriggle (this requires a cool head), –Drive (repeat).

Proof: Trotters Product Formula

Linear Systems

$$\dot{x} = Ax + Bu = f(x) + g(x)u$$



$$\begin{aligned} [f,g] &= [Ax, B] = 0 - AB \\ [g,[f,g]] &= 0 \\ [f,[f,g]] &= [Ax, -AB] = A^2B \\ &\vdots \\ \text{Ad}_f^k g &= [f,[f,\ldots,[f,g]]] = (-1)^k A^k B \end{aligned}$$

Controllability Theorems

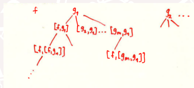
$$\dot{x} = f(x) + \sum_i g_i(x) u_i$$

Let $A(x_0)$ be the reachable set from x_0, i.e. all points that can be reached from x_0 using a suitable control u

Accessibility The system has the accessibility property at x if $A(x)$ has nonempty interior

Accessibility theorem

$$C \quad = \quad \text{smallest Lie subalg. containing } \{f, g_1, \ldots, g_m\}$$



Theorem If for all x_0 the Lie-bracket tree contains n linearly independent elements, then the system is has the accessbility property

$$\dim C = n \implies \text{can reach open set}$$

If $f = 0$, (or more generally $f(x,u)$ is "symmetric", see Glad) then the system is controllable: $A(x_0) = R^n$

Reading Assignment

For a precis formulation, and more about "controllability" vs "accessability" see

T. Glad, Nonlinear Control Theory, Chapter 8, pp 73-81