Synthesis, Nonlinear design

- Introduction
- Relative degree & zero-dynamics (rev.)
- Exact Linearization (intro)
- Control Lyapunov functions
- Lyapunov redesign
- Nonlinear damping
- Backstepping
  - Control Lyapunov functions (CLFs)
  - passivity
  - robust/adaptive

Ch 13.1-13.2, 14.1-14.3 Nonlinear Systems, Khalil

The Joy of Feedback, P V Kokotovic

Why nonlinear design methods?

- Linear design degraded by nonlinearities (e.g. saturations)
- Linearization not controllable (e.g. pocket parking)
- Long state transitions (e.g. satellite orbits)
- Inherently nonlinear...

Relative degree

" A system's relative degree: How many times you need to take the derivative of the output signal before the input shows up"

Note: A nonlinear system may have state-dependent relative degree.

Example: The ball and beam process (see process homepage for more information).

If nothing else stated we assume a fixed relative degree in the sequel.

For a nonlinear system with *relative degree* d

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{1}$$

we have

$$\begin{aligned} \dot{y} &= \frac{d}{dt}h(x) = \frac{\partial h(x)}{\partial x}\dot{x} = \frac{\partial h}{\partial x}f(x) + \frac{\partial h}{\partial x}g(x)u \\ &= L_f h(x) + \underbrace{L_g h(x)}_{=0 \; if \;\, d>1} u \\ &\vdots \\ y^{(k)} &= L_f^k h(x) \qquad \text{if } k < d \\ &\vdots \\ y^{(d)} &= L_f^d h(x) + L_g L_f^{(d-1)} h(x) u \end{aligned} \tag{2}$$

Using the same kind of coordinate transformations as for the feedback linearizable systems above, we can introduce new state space variables, ξ, where the first d coordinates are chosen as

$$\begin{cases} \xi_1 &= h(x) \\ \xi_2 &= L_f h(x) \\ \vdots & \\ \xi_d &= L_f^{(d-1)} h(x) \end{cases} \tag{3}$$

Under some conditions on involutivity, the Frobenius theorem guarantees the existence of another $(n-d)$ functions to provide a local state transformation of full rank. Such a coordinate change transforms the system to the *normal form*

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ &\vdots \\ \dot{\xi}_{d-1} &= \xi_d \\ \dot{\xi}_d &= L_f^d h(\xi, z) + L_g L_f^{d-1} h(\xi, z) u \\ \dot{z} &= \psi(\xi, z) \\ y &= \xi_1 \end{aligned} \tag{4}$$

where $\dot{z} = \psi(\xi, z)$ represent the zero dynamics of order $n-d$ [Byrnes+Isidori 1991].

Example (Zero dynamics for linear systems)

Consider the linear system

$$y = \frac{s-1}{s^2+2s+1}u \tag{5}$$

with the following state-space description

$$\begin{cases} \dot{x}_1 &= -2x_1 + x_2 & +u \\ \dot{x}_2 &= -x_1 & -u \\ y &= x_1 \end{cases} \tag{6}$$

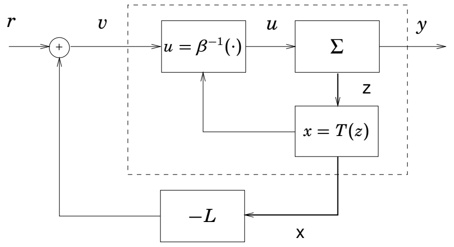
We have the relative degree =1
Find the zero-dynamics, by assigning $y \equiv 0$.

$$\begin{aligned} &y \equiv 0 \Rightarrow x_1 \equiv 0 \;\Rightarrow\; \dot{x}_1 \equiv 0 \;\Rightarrow\; x_2 + u = 0 \\ &\Rightarrow \dot{x}_2 = -u = x_2 \end{aligned} \tag{7}$$

The remaining dynamics is an unstable system corresponding to the zero $s = 1$ in the transfer function (5).

Exact (feedback) Linearization

Idea: Transform the nonlinear system into a linear system by means of feedback and/or a change of variables. After this, a stabilizing state feedback is designed.



Inner *feedback linearization* and outer *linear* feedback control

For general nonlinear systems, feedback linearization comprises

- state transformation
- inversion of nonlinearities
- linear feedback

Simple example

$$\ddot{x} = \frac{g}{l}\sin(x) + \cos(x)u$$

Put

$$u = \frac{1}{\cos(x)}(-\frac{g}{l}\sin(x) + v)$$

gives (locally)

$$\ddot{x} = v$$

Design linear controller $v = -l_1 x + -l_2 \dot{x}$, etc

State transformation

"More difficult" example, where we need a state transformation

$$\begin{aligned} \dot{x}_1 &= a\sin(x_2) \\ \dot{x}_2 &= -x_1^2 + u \end{aligned}$$

Can not cancel $a\sin(x_2)$. Introduce

$$\begin{aligned} z_1 &= x_1 \\ z_2 &= a\sin x_2 \end{aligned}$$

so that

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= (-z_1^2 + u)a\cos x_2 \end{aligned}$$

Then feedback linearization is (locally) possible by

$$u = z_1^2 + v/(a\cos(z_2))$$

Feedback linearization ("nonlinear version of pole-zero cancellation")

Feedback linearization can be interpreted as a nonlinear version of pole-zero cancellations which can not be used if the zero-dynamics are unstable, i. e., for *nonminimum-phase system*.

Linear systems: See paper [Middleton (1999) Automatica 35(5), "Slow stable open-loop poles: to cancel or not to cancel"]

When to cancel nonlinearities?

$$\begin{aligned} \dot{x}_1 &= -x_1^3 + u_1 \\ \dot{x}_2 &= x_2^3 + u_2 \end{aligned} \tag{8}$$

Nonrobust and/or not necessary.
However, note the difference between tracking or regulation!!

Will see later how "optimal criteria" will give hints.

"Matching" uncertainties

$$\begin{aligned} \dot{x}_1 &= x_2 \\ &\vdots \\ \dot{x}_{n-1} &= x_d \\ \dot{x}_n &= L_f^d h(x,z) + L_g L_f^{d-1} h(x,z)u \\ \dot{z} &= \psi(x,z) \\ y &= x_1 \end{aligned} \tag{9}$$

Integrator chain and nonlinearities (+ zero-dynamics)
Note that uncertainties due to parameters etc. are "collected in"

$$L_f^d h(x,z) + L_g L_f^{d-1} h(x,z)u$$

Achieving passivity by feedback (*Feedback passivation*)
Need to have

- relative degree one
- weakly minimum phase

NOTE! (Nonlinear) relative degree and zero-dynamics *invariant* under feedback!
Two major challenges:

- avoid non-robust cancellations
- make it constructive by finding matching input-output pairs

Exact Linearization

- Often useful in simple cases
- Important intuition may be lost
- Nonlinear version of "pole-zero cancellations"
- Related to "Lie brackets" and "flatness"
- Known under several different names, e.g.,
  - Computed torque (robotics)

From analysis to synthesis

Lyapunov criterion Search for (V, u) such that

$$\frac{\partial V}{\partial x}[f + gu] < 0$$

IQC criterion Search for $Q(s)$ and $\tau_1, \dots, \tau_m$ such that

$$\begin{bmatrix} [T_1 + T_2QT_3](i\omega) \\ I \end{bmatrix}^* \left[\sum_k \tau_k \Pi_k(i\omega)\right] \begin{bmatrix} [T_1 + T_2QT_3](i\omega) \\ I \end{bmatrix} < 0$$

for $\omega \in [0, \infty]$

In both cases, the problem is non-convex and hard.
Heuristic idea: Iterate between the arguments

Convexity for state feedback

Problem Suppose $\alpha \le \phi(v)/v \le \beta$. Given the system

$$\dot{x} = f_u(x) := Ax + E\phi(Fx) + Bu$$

find $u = -Lx$ and $V(x) = x^T P x$ such that $\frac{\partial V}{\partial x} f_u(x) < 0$

Solution Solve for P, L

$$(A + \alpha EF - BL)^T P + P(A + \alpha EF - BL) < 0$$
$$(A + \beta EF - BL)^T P + P(A + \beta EF - BL) < 0$$

or equivalently convex in $(Q, K) = (P^{-1}, LP^{-1})$

$$(AQ + \alpha EFQ - BK)^T + (AQ + \alpha EFQ - BK) < 0$$
$$(AQ + \beta EFQ - BK)^T + (AQ + \beta EFQ - BK) < 0$$

Control Lyapunov Function (CLF)

A positive definite radially unbounded C^1 function V is called a CLF for the system $\dot{x} = f(x, u)$ if for each $x \neq 0$, there exists u such that

$$\frac{\partial V}{\partial x}(x) f(x, u) < 0 \qquad \text{(Notation: } L_f V(x) < 0)$$

When $f(x, u) = f(x) + g(x)u$, V is a CLF if and only if

$$L_f V(x) < 0 \text{ for all } x \neq 0 \text{ such that } |L_g V(x)| = 0$$

Example

Check if $V(x, y) = [x^2 + (y + x^2)^2]^2/2$ is a CLF for the system

$$\begin{cases} \dot{x} = xy \\ \dot{y} = -y + u \end{cases}$$

$$L_f V(x, y) = x^2 y + (y + x^2)(-y + 2x^2 y)$$
$$L_g V(x, y) = 2(y + x^2)[x^2 + (y + x^2)^2]$$

$$L_g V(x, y) = 0 \quad \Rightarrow \quad y = -x^2 \quad \Rightarrow \quad L_f V(x, y) = -x^4 < 0 \quad \text{if } (x, y) \neq$$

Sontag's formula

If V is a CLF for the system $\dot{x} = f(x) + g(x)u$, then a continuous asymptotically stabilizing feedback is defined by

$$u(x) := \begin{cases} 0 & \text{if } L_g V(x) = 0 \\ -\frac{L_f V + \sqrt{(L_f V)^2 + ((L_g V)(L_g V)^T)^2}}{(L_g V)(L_g V)^T}[L_g V]^T & \text{if } L_g V(x) \neq 0 \end{cases}$$

Note: Can cancel factor $L_g V \neq 0$ if scalar.

$$u(x) := \begin{cases} 0 & \text{if } L_g V(x) = 0 \\ -\frac{L_f V + \sqrt{(L_f V)^2 + (L_g V)^4}}{L_g V}(x) & \text{if } L_g V(x) \neq 0 \end{cases}$$

Backstepping idea

Problem

Given a CLF for the system

$$\dot{x} = f(x, u)$$

find one for the extended system

$$\dot{x} = f(x, y)$$
$$\dot{y} = h(x, y) + u$$

Idea

Use y to control the first system. Use u for the second.

Note potential for recursivity

Motivation: Feedback Linearization

One of the drawbacks with feedback linearization is that exact cancellation of nonlinear terms may not be possible due to e. g., parameter uncertainties.

A suggested solution:

- stabilization via feedback linearization around a nominal model
- consider known bounds on the uncertainties to provide an additional term for stabilization (*Lyapunov redesign*)

Lyapunov Redesign

Consider the nominal system

$$\dot{x} = f(x, t) + G(x, t)u$$

with the known control law

$$u = \psi(x, t)$$

so that the system is uniformly asymptotically stable.

Assume that a Lyapunov function $V(x, t)$ is known s.t.

$$\begin{aligned} \alpha_1(||x||) \le V(x, t) &\le \alpha_2(||x||) \\ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}[f(t, x) + G\psi] &\le -\alpha_3(||x||) \end{aligned}$$

Lyapunov Redesign — *cont.*

Perturbed system

$$\dot{x} = f(x,t) + G(x,t)[u + \delta] \qquad (10)$$

disturbance $\delta = \delta(t, x, u)$

Assume the disturbance satisfies the bound

$$||\delta(t, x, \psi + v)|| \leq \rho(x,t) + \kappa_0 ||v||$$

If we know ρ and κ_0 how do we design *additional control* v such that $u = \psi(x,t) + v$ stabilizes (10)?

The **matching condition**: perturbation enters at **same** place as control signal u.

Apply $u = \psi(x,t) + v$

$$\dot{x} = f(x,t) + G(x,t)\psi + G(x,t)[v + \delta(t, x, \psi + v)] \qquad (11)$$

$$\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}[f(t,x) + G\psi] + \frac{\partial V}{\partial x} G[v + \delta] \leq$$
$$-\alpha_3(||x||) + \frac{\partial V}{\partial x} G[v + \delta]$$

Introduce $w = [\frac{\partial V}{\partial x} G]$

$$\dot{V} \leq -\alpha_3(||x||) + w^T v + w^T \delta$$

Choose v such that $w^T v + w^T \delta \leq 0$:

Two alternatives presented in Khalil ($|| \cdot ||_2$-norm / $|| \cdot ||_\infty$-norm)

Note: v appears at same place as δ due to the matching condition

Lyapunov Redesign — *cont.*

$$w^T v + w^T \delta \leq w^T v + ||w^T||_2 ||\delta||_2$$
$$w^T v + w^T \delta \leq w^T v + ||w^T||_1 ||\delta||_\infty$$

Alternative 1:

If

$$||\delta(t, x, \psi + v)||_2 \leq \rho(x,t) + \kappa_0 ||v||_2, \quad 0 \leq \kappa_0 < 1$$

take

$$v = -\eta(t,x) \frac{w}{||w||_2}$$

where $\eta \geq \rho / (1 - \kappa_0)$

Alternative 2:

If

$$||\delta(t, x, \psi + v)||_\infty \leq \rho(x,t) + \kappa_0 ||v||_\infty, \quad 0 \leq \kappa_0 < 1$$

take

$$v = -\eta(t,x)\, \mathrm{sgn}\, w$$

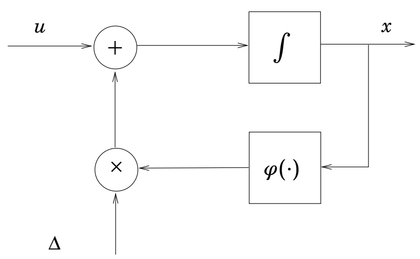
where $\eta \geq \rho / (1 - \kappa_0)$

Restriction on $\kappa_0 < 1$ but not on growth of ρ.

Alt 1 and alt 2 coincide for single-input systems.

Note: control laws are discontinues fcn of x (risk of *chattering*)

Example: Matched uncertainty



$$\dot{x} = u + \varphi(x)\Delta(t)$$

Example *cont.*

Example:

Exponentially decaying disturbance $\Delta(t) = \Delta(0)e^{-kt}$

linear feedback $u = -cx, \quad c > 0$

$\varphi(x) = x^2$

$$\dot{x} = -cx + \Delta(0)e^{-kt}x^2$$

Similar to peaking problem in the first lecture: Finite escape of solution to infinity if $\Delta(0)x(0) > c + k$

We want to guarantee that x(t) stay bounded for all initial values $x(0)$ and all bounded disturbances $\Delta(t)$

Nonlinear damping

Modify the control law in the previous example as:

$$u = -cx - s(x)x$$

where

$$-s(x)x$$

will be denoted *nonlinear damping*.

Use the Lyapunov function candidate $V = \dfrac{x^2}{2}$

$$\dot{V} = xu + x\varphi(x)\Delta$$
$$= -cx^2 - x^2 s(x) + x\varphi(x)\Delta$$

How to proceed?

Choose

$$s(x) = \kappa\varphi^2(x)$$

to complete the squares!

$$\dot{V} = -cx^2 - x^2 s(x) + x\varphi(x)\Delta$$
$$= -cx^2 - \kappa\left[x\varphi - \frac{\Delta}{2\kappa}\right]^2 + \kappa \cdot \frac{\Delta^2}{4\kappa^2} \quad \leq -cx^2 + \frac{\Delta^2}{4\kappa}$$

Note! $\dot{V}$ is negative whenever

$$|x(t)| \geq \frac{\Delta}{2\sqrt{\kappa c}}$$

Can show that $x(t)$ converges to the set

$$R = \left\{ x : |x(t)| \le \frac{\Delta}{2\sqrt{\kappa c}} \right\}$$

i. e., x(t) stays bounded for all bounded disturbances Δ

Remark: The nonlinear damping $-\kappa x \varphi^2(x)$ renders the system Input-To-State Stable (ISS) with respect to the disturbance.

Young's inequality

Let $p > 1,\ q > 1$ s.t. $(p-1)(q-1) = 1$, then for all $\epsilon > 0$ and all $(x,\ y) \in |R^2$

$$xy < \frac{\epsilon^p}{p}|x|^p + \frac{1}{q\epsilon^q}|y|^q$$

Standard case: $(p = q = 2,\ \epsilon^2/2 = \kappa)$

$$xy < \kappa|x|^2 + \frac{1}{4\kappa}|y|^2$$

Our example:

$$x\varphi(x)\Delta(t) < \kappa x^2\varphi^2(x) + \frac{\Delta^2(t)}{4\kappa}$$

Backstepping idea

Problem

Given a CLF for the system

$$\dot{x} = f(x, u)$$

find one for the extended system

$$\begin{aligned} \dot{x} &= f(x, y) \\ \dot{y} &= h(x, y) + u \end{aligned}$$

Idea

Use y to control the first system. Use u for the second.

Note: potential for recursivity

Backstepping

Let V_x be a CLF for the system $\dot{x} = f(x) + g(x)\bar{y}$ with corresponding asymptotically stabilizing control law $\bar{y} = \phi(x)$. Then $V(x, y) = V_x(x) + [y - \phi(x)]^2/2$ is a CLF for the system'

$$\begin{aligned} \dot{x} &= f(x) + g(x)y \\ \dot{y} &= h(x, y) + u \end{aligned}$$

with corresponding control law

$$u = \frac{\partial \phi}{\partial x}[f(x) + g(x)y] - \frac{\partial V_x}{\partial x}g(x) - h(x, y) + \phi(x) - y$$

Proof.

$$\begin{aligned} \dot{V} &= (\partial V_x/\partial x)(f + gy) + (y - \phi)\,[h + u - (\partial\phi/\partial x)\cdot(f + gy)] \\ &= (\partial V_x/\partial x)(f + g\phi) + (y - \phi)[(\partial V_x/\partial x)g - (\partial\phi/\partial x)\cdot(f + gy) + h \\ &= (\partial V_x/\partial x)(f + g\phi) - (y - \phi)^2 < 0 \end{aligned}$$

Backstepping Example

For the system

$$\begin{cases} \dot{x} = x^2 + y \\ \dot{y} = u \end{cases}$$

we can choose $V_x(x) = x^2$ and $\phi(x) = -x^2 - x$ to get the control law

$$\begin{aligned} u &= \phi'(x)f(x, y) - h(x, y) + \phi(x) - y \\ &= -(2x + 1)(x^2 + y) - x^2 - x - y \end{aligned}$$

with Lyapunov function

$$\begin{aligned} V(x, y) &= V_x(x) + [y - \phi(x)]^2/2 \\ &= x^2 + (y + x^2 + x)^2/2 \end{aligned}$$

Example again (step by step)

$$\begin{cases} \dot{x}_1 = {x_1}^2 + x_2 \\ \dot{x}_2 = u(x) \end{cases} \tag{12}$$

Find u(x) which stabilizes (12).

Idea : Try first to stabilize the x_1-system with x_2 and then stabilize the whole system with u.

We know that if $x_2 = -x_1 - x_1^2$
then $x_1 \to 0$ asymptotically (exponentially)
as $t \to \infty$.

We can't expect to realize $x_2 = \alpha(x_1)$ exactly, but we can always try to get

t

he error $\to 0$.

Introduce the error states

$$\begin{cases} z_1 = x_1 \\ z_2 = x_2 - \alpha_1(x_1) \end{cases} \tag{13}$$

where $\alpha_1(x_1) = -x_1 - x_1^2$

$$\begin{aligned} \Rightarrow \dot{z}_1 &= \dot{x}_1 = z_1^2 + \overbrace{z_2 + \alpha_1(z_1)}^{x_2} = \\ &= z_1^2 + z_2 - z_1^2 - z_1 = -z_1 + z_2 \\ \dot{z}_2 &= \dot{x}_2 - \dot{\alpha}_1 = u(x) - \overbrace{\dot{\alpha}_1}^{known} \\ \dot{\alpha}_1 &= \frac{d}{dt}(-z_1^2 - z_1) = -z_1\dot{z}_1 - \dot{z}_1 \\ &= -z_1(-z_1 + z_2) - (-z_1 + z_2) = \\ &= z_1^2 - z_1 z_2 - z_2 - z_1 \end{aligned}$$

Start with a Lyapunov for the first subsystem (z_1-dynamics):

$$\begin{aligned} V_1 &= \frac{1}{2}z_1^2 \ge 0 \\ \dot{V}_1 &= z_1\dot{z}_1 = -z_1^2 + z_1 z_2 \end{aligned}$$

Note :
If $z_2 = 0$ we would achieve $V_1 = -z_1^2 \le 0$
with $\alpha_1(x_1)$

Now look at the augmented Lyapunov fcn for the error system

$$
\begin{aligned}
V_2 &= V_1 + \frac{1}{2} z_2^2 \geq 0 \\
\dot{V}_2 &= \dot{V}_1 + z_2 \dot{z}_2 = \\
&= -z_1^2 + z_1 z_2 + z_2 (u - z_1^2 + z_1 z_2) \\
&= -z_1^2 + z_2 \underbrace{(u - z_1^2 + z_1 z_2 + z_2 + z_1)}_{choose\ = -z_2} \\
&= -z_1^2 - z_2^2 \leq 0
\end{aligned}
$$

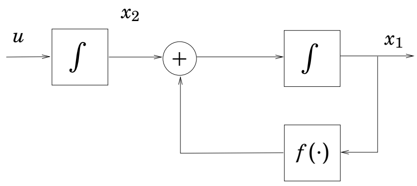
so if $u = z_1^2 - z_1 z_2 - z_2 - z_1$

$\Rightarrow (z_1, z_2) \rightarrow 0$ asymptotically (exponentially)

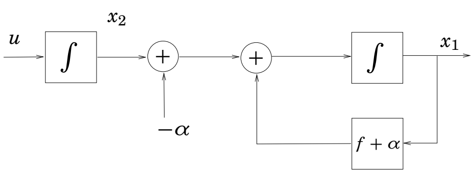
$\Rightarrow (x_1, x_2) \rightarrow 0$ asymptotically

As $z_1 = x_1$ and $z_2 = x_2 - \alpha_1 = x_2 + x_1^2 + x_1$,
we can express u as a (nonlinear) state feedback function of x_1 and x_2.

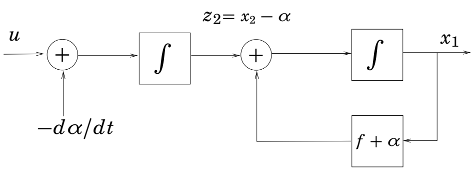
Backward propagation of desired control signal



If we could use x_2 as control signal, we would like to assign it to $\alpha(x_1)$ to stabilize the x_1-dynamics.



Move the control “backwards” through the integrator



Note the change of coordinates!

Adaptive Backstepping

System :

$$
\begin{cases}
\dot{x}_1 = x_2 + \theta \gamma(x_1) \\
\dot{x}_2 = x_3 \\
\dot{x}_3 = u(t)
\end{cases}
\tag{14}
$$

where γ is a known function of x_1 and
θ is an unknown parameter
Introduce new (error) coordinates

$$
\begin{cases}
z_1(t) = x_1(t) \\
z_2(t) = x_2(t) - \alpha_1(z_1, \hat{\theta})
\end{cases}
\tag{15}
$$

where α_1 is used as a control to stabilize the z_1- system w.r.t a certain Lyapunov-function.

Lyapunov function : $V_1 = \frac{1}{2} {z_1}^2 + \frac{1}{2} \tilde{\theta}^2$ where $\tilde{\theta} = (\hat{\theta} - \theta)$ is the parameter error

(Back-) Step 1:

$$
\begin{aligned}
\dot{z}_1(t) &= \overbrace{z_2(t) + \alpha_1(z_1, \hat{\theta})}^{x_2} + \theta \gamma(z_1(t)) \\
\dot{V}_1 &= z_1 \dot{z}_1 + \tilde{\theta} \dot{\hat{\theta}} = z_1 (z_2 + \alpha_1 + \theta \gamma) + \tilde{\theta} \dot{\hat{\theta}} = \\
&= z_1 [\, z_2 + \underbrace{\alpha_1 + \hat{\theta} \gamma}_{-z_1} \,] + \tilde{\theta} (\dot{\hat{\theta}} - \underbrace{z_1 \gamma}_{\tau_1})
\end{aligned}
$$

Choose $\alpha_1 = -z_1 - \hat{\theta} \gamma$

$$
\Rightarrow \dot{V}_1 = -z_1^2 + z_1 z_2 + \tilde{\theta} (\dot{\hat{\theta}} - \tau_1)
$$

Note: If we used $\dot{\hat{\theta}} = \tau_1$ as update law
and if $z_2 = 0$ then $\dot{V}_1 = -z_1^2 \leq 0$

Step 2: Introduce $z_3 = x_3 - \alpha_2(z_1, z_2, \hat{\theta})$ and use α_2 as control to stabilize the (z_1, z_2)-system

etc.

Observer backstepping

Observer backstepping is based on the following steps:

1. A (nonlinear) observer is designed which provides (exponentially) convergent estimates.
2. Backstepping is applied to a system where the states have been replaces by their estimates.
   The observation errors are regarded as (bounded) disturbances and handled by *nonlinear damping.*

Backstepping applies to systems in *strict-feedback form*

$$
\begin{aligned}
\dot{x}_1 &= f_1(x_1) + x_2 \\
\dot{x}_2 &= f_2(x_1, x_2) + x_3 \\
&\vdots \\
\dot{x}_n &= f_n(x_1, x_2, \ldots x_{n-1}, x_n) + u
\end{aligned}
$$

Compare with
Strict-feedforward systems

$$
\begin{aligned}
\dot{x}_1 &= x_2 + f_1(x_2, x_3, \ldots, x_n, u) \\
\dot{x}_2 &= x_3 + f_2(x_3, \ldots, x_n, u) \\
&\vdots \\
\dot{x}_{n-1} &= x_n + f_{n-1}(x_n, u) \\
\dot{x}_n &= u
\end{aligned}
$$

Nonlinear Control Synthesis *cont'd*

- Optimal and inverse(!) optimal design
- Saturated control and feedforwarding

Synthesis *cont'd*

- HJB
- Inverse optimal control
- Stabilization with Saturations
- Integrator forwarding
- Relations between the concepts

Optimality

Two main alternatives

- Pontryagin's Maximum Principle (Necessary cond)
- Hamilton-Jacobi-Bellman (Dyn prog.) (Sufficient cond)

Based on Ch 18.5 [Glad & Ljung]

Find the state feedback law

$$u = k(x)$$

which solves minimization problem

$$\begin{aligned} &\min_u \int_{t_0}^{t_f} (L(x, u)dt + \varphi(t_f, x(t_f)) \\ &\dot{x} = f(x(t), u(t)) \\ &u \in \mathcal{U}, \quad t_0 \le t \le t_f \\ &x(t_0) = x_0, \quad \psi(t_f, x(t_f)) = 0 \end{aligned}$$

Assume that u^* and x^* solves this optimization problem.

Define $V(t_0, x_0)$ as the optimal return function

$$V(t_0, x_0) = \int_{t_0}^{t_f} (L(x^*, u^*)dt + \varphi(t_f, x^*(t_f))$$

if we start in $(t_0, x(t_0) = x_0)$

Remark: Need to satisfy ...

Property of V:

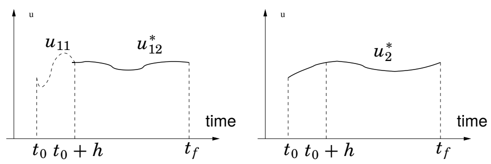
If V is differentiable along a solution $x(t)$, then

$$\frac{d}{dt} V(t, x(t)) + L(x(t), u(t)) \ge 0 \qquad (16)$$

with equality for x^* and u^*.

Assume that we for

- $t \in [t_0, t_0 + h]$ use any control $u(t)$
- $t \in [t_0 + h, t_f]$ use optimal control $u(t)^*$



The 0ptimization criterion becomes

$$\int_{t_0}^{t_0+h} (L(x(r), u(r))dr + V(t_0 + h, x(t_0 + h))$$

If optimal control from t_0: $V(t_0, x(t_0)) \Longrightarrow$

$$V(t_0, x(t_0)) \le \int_{t_0}^{t_0+h} (L(x(r), u(r))dr + V(t_0 + h, x(t_0 + h))$$

which gives

$$\frac{V(t_0 + h, x(t_0 + h)) - V(t_0, x(t_0))}{h} + \frac{1}{h} \int_{t_0}^{t_0+h} (L(x(r), u(r))dr \ge 0$$

which in the limit $h \to 0^+$ gives

$$\frac{d}{dt} V(t, x(t)) + L(x(t), u(t)) \ge 0$$

Theorem: If the *optimal return value* V is differentiable it satisfies

$$-\frac{\partial V}{\partial t} = \min_{u \in \mathcal{U}} \left(\frac{\partial V}{\partial x} f(x, u) + L(x, y) \right) \qquad (17)$$

Proof: The chain rule gives

$$\frac{d}{dt} V(t, x(t)) = V_t + V_x f$$

and from Eq.(16) gives

$$-\frac{\partial V}{\partial t} \le \frac{\partial V}{\partial x} f(x, u) + L(x, y)$$

with equality for optimal control u^*.

Eq.(17) is called

the Hamilton-Jacobi equation (HJ) for a finite t_f

the Hamilton-Jacobi-Bellman equation (HJB) for $t_f = \infty$.

Remarks: Severe restriction to assume V differentiable (*e.g.,* bang-bang solutions for minimal time problems give "corners" in V but results can be extended to this case as well.

- State feedback law $u = k(t,\ x) = \arg\min_{u\in\mathcal{U}}\left(\frac{\partial V}{\partial x}f(x,u) + L(x,y)\right)$
- Necessary conditions while Pontryagin gives sufficient.

Consider the system

$$\dot{x} = f(x) + g(x)u$$

Find $u = u^*$ such that

(i) u achieves asymptotic stability of the origin $x = 0$

(ii) u minimizes the cost functional

$$\int_0^\infty (l(x) + u^T R(x)u)dt \tag{18}$$

where $l(x) \geq 0$ and $R(x) \geq 0 \forall x$.

For a given optimal feedback $u(x)^*$ the value of V depends on the initial state $x(0)$: $V(x(0))$ or simply $V(x)$ (and start time according to previous slides).

Theorem (Optimality and Stability)
Suppose there exist a $\mathcal{C}^1$-function $V(x) \geq 0$ which satisfies the *Hamilton-Jacobi-Bellman equation*

$$\begin{aligned} l(x) + L_f V(x) - \frac{1}{4}L_g V(x)R^{-1}(L_g V(x))^T &= 0 \\ V(0) &= 0 \end{aligned} \tag{19}$$

such that the feedback control

$$u^*(x) = -\frac{1}{2}R^{-1}(L_g V(x))^T$$

achieves asymptotic stability of the origin $x = 0$.

Then $u^*(x)$ is the *optimal stabilizing control* which minimizes the cost (18).

Example:
Linear system

$$\dot{x} = Ax + Bu$$

Cost Function

$$V = \int_0^\infty (x^T C^T Cx + u^T Ru)dt, \qquad R > 0$$

Riccati-equation

$$PA + AP^T - PBR^{-1}B^T P + C^T C = 0 \tag{20}$$

If (A,B) controllable and (A,C) observable, then (20) has a unique solution $P = P^T > 0$ such that the optimal cost is $V = x^T P x$ and

$$u^*(x) = -R^{-1}B^T P x$$

is the optimal stabilizing control

5-min exercise:
Consider the system

$$\dot{x} = x^2 + u$$

and the cost functional

$$V = \int_0^\infty (x^2 + u^2)dt$$

What is the optimal stabilizing control?

HJB:

$$x^2 + \frac{\partial V}{\partial x}x^2 - \frac{1}{4}\left(\frac{\partial V}{\partial x}\right)^2 = 0, \qquad V(x) = 0$$

$$\begin{aligned} \frac{\partial V}{\partial x} &= 2x^2 \pm \sqrt{4x^4 + 4x^2} \\ &= 2x^2 + 2x\sqrt{x^2+1} \end{aligned} \tag{21}$$

$$V(x) = \frac{2}{3}x^3 + \frac{2}{3}(x^2+1)^{3/2} + C, \qquad C = -2/3 \text{ so that } V(0) = 0 \tag{22}$$

$$u^*(x) = -\frac{1}{2}\frac{\partial V}{\partial x} = -x^2 - x\sqrt{x^2+1}$$

Remark: We have chosen the positive solution in (21) as $V(x) \geq 0$

Remark: If (A,B) stabilizable and (A,C) detectable then P is positive semi-definite.
Example (non-detectability in cost)

System

$$\dot{x} = x + u$$

Cost functional

$$V = \int_0^\infty u^2 dt$$

Riccati-eq

$$2P - P^2 = 0, \qquad P = 0 \text{ or } P = 2$$

Corresponding HJB

$$x\frac{\partial V}{\partial x} - \frac{1}{4}(\frac{\partial V}{\partial x})^2 = 0, \qquad V(0) = 0$$

$$V = 0 \text{ or } V = 2x^2$$

Inverse optimality

A stabilizing control law $u(x)$ solves an *inverse* optimal problem for the system

$$\dot{x} = f(x) + g(x)u$$

if it can be written as

$$u(x) = -k(x)/2 = -\frac{1}{2}R^{-1}(x)(L_gV(x))^T, \qquad R(x) > 0$$

where $V(x) \geq 0$ and

$$\dot{V} = L_fV + L_gV = L_fV \underbrace{- \frac{1}{2}L_gVk(x)}_{-l(x)} \leq 0$$

Then $V(x)$ is the solution of the HJB-eqn

$$l(x) + L_fV - \frac{1}{4}(L_gV)R^{-1}(L_gV)^T = 0$$

The underlying idea of formulating an *inverse* optimal problem is to get some **help to avoid non-robust cancellations** and gain some stability margins.

Example: Non-robust cancellation
Consider the system

$$\dot{x} = x^2 + u$$

and the control law

$$u_n = -x^2 - x \qquad \Rightarrow \qquad \dot{x} = -x$$

However, if there is some small perturbation gain $u = (1+\epsilon)u_n$, we get

$$\dot{x} = -(1+\epsilon)x - \epsilon x^2$$

This system may has finite escape time solutions.

How does u^* from previous example behave?

Damping Control / Jurdjevic-Quinn

Consider the system

$$\dot{x} = f(x) + g(x)u$$

Assume that the *drift part* of the system is stable, *i.e.,*

$$\dot{x} = f(x), \quad f(0) = 0$$

and that we know a function $V(x)$ such that $L_fV \leq 0$ for all x.
How to make it asymptotically stable (robustly)?

To add more damping to the system to render it asymptotically stable the following suggestion was made by Jurdjevic-Quinn (1978)

$$\dot{V} = L_fV + L_gVu \leq L_gVu$$

Choose

$$u = -\kappa \cdot (L_gV)^T$$

It also solves the global optimization problem for the cost functional

$$V(x) = \int_0^\infty (l(x) + \frac{2}{\kappa}u^Tu)dt$$

for the state cost function

$$l(x) = -L_fV + \frac{\kappa}{2}(L_gV)(L_gV)^T \geq 0$$

Connection to passivity:
The system

$$\dot{x} = f(x) + g(x)u$$
$$y = (L_gV)^T(x)$$

is passive with $V(x)$ as storage function if $L_fV \leq 0$ as

$$\dot{V} = L_fV + L_gVu \leq y^Tu$$

The feedback law $u = -\kappa y$ guarantees GAS if the system is ZSD (zero state detectable).

Note: May be a conservative choice as it does not fully exploit the possibility to choose $V(x)$ for the whole system (only $\dot{x} = f(x)$).

Systems with saturations of control signal

Problem: System runs in "open loop" when in saturation

- Anti-windup designs from FRTN05
- Consider Lyapunov function candidates of type $V = log(1 + x^2)$ (see Lecture 1)
- Saturated controls [Sussmann, Yang And Sontag]
- Cascaded saturations [Teel *et al*]

Feedforward systems

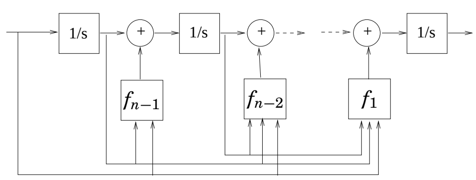
Particular form of cascaded systems

1991 **A. Teel**
... Sussman, Sontag, Yang
... Saberi, Lin

1996 Mazenc, Praly
1996 Sepulchre, Jankovic, Kokotovic

Strict-feedforward systems

$$\begin{aligned}
\dot{x}_1 &= x_2 + f_1(x_2, x_3, \ldots, x_n, u)\\
\dot{x}_2 &= x_3 + f_2(x_3, \ldots, x_n, u)\\
&\vdots\\
\dot{x}_{n-1} &= x_n + f_{n-1}(x_n, u)\\
\dot{x}_n &= u
\end{aligned}$$



Compare with e.g.
Strict-feedback systems

$$\begin{aligned}
\dot{x}_1 &= x_2 + f_1(x_1)\\
\dot{x}_2 &= x_3 + f_2(x_1, x_2)\\
&\vdots\\
\dot{x}_n &= x_n + f_n(x_1, x_2, \dots x_{n-1}) + u
\end{aligned}$$

Strict-feedforward systems are, in general, **not** feedback linearizable!
(*i.e.,* neither exact linearization nor backstepping is applicable for stabilization)

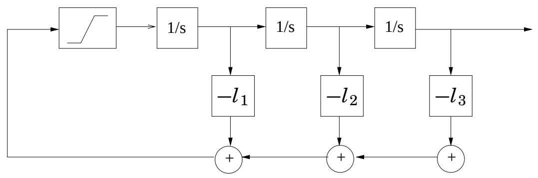
Restriction: Does not cover systems of the type

$$\begin{aligned}
&\dots\\
&\dot{x}_k = -x_k^2 + \dots\\
&\dots
\end{aligned}$$

i.e. don't have to worry about

finite escape-time

Sussman and Yang (1991) :
There does not exist any (simple) saturated feedback-law which stabilizes an integrator chain of order ≥ 3 globally.



Teel's idea:
using nested saturations

$$u = -\sigma_n(h_n(x) + \sigma_{n-1}(h_{n-1}(x) + \dots + \sigma_1(h_1(x))\dots)$$

Definition: σ is a *linear saturation* for (L, M) if

- σ is continuous and nondecreasing
- $\sigma(s) = s$ when $|s| \leq L$
- $|\sigma(s)| \leq M$, $\forall s \in R$

Theorem (Teel):
For an integrator chain of any order and for any set $\{(L_i, M_i)\}$ where $L_i \leq M_i$ and $M_i < \frac{1}{2}L_{i+1}$, there exists $\{h_i\}$ for all linear saturations $\{\sigma_i\}$ such that the bounded control

$$u = -\sigma_n(h_n(x) + \sigma_{n-1}(h_{n-1}(x) + \dots + \sigma_1(h_1(x))\dots)$$

results in global asymptotic stability for the closed loop system.

Sketch of proof: (n=3, $L_i = M_i$)
Consider a state transformation $y = Tx$ which transforms the integrator chain into

$$\dot{y} = Ay + Bu$$

where

$$A = \begin{pmatrix} 0 & 1 & 1\\ 0 & 0 & 1\\ 0 & 0 & 0 \end{pmatrix},\ B = \begin{pmatrix} 1\\ 1\\ 1 \end{pmatrix}$$

The control law

$$u = -\sigma_3(y_3 + \sigma_2(y_2 + \sigma_1(y_1)))$$

will give the closed loop system

$$\begin{aligned}
\dot{y}_1 &= y_2 + y_3 && -\sigma_3(y_3 + \sigma_2(y_2 + \sigma_1(y_1)))\\
\dot{y}_2 &= y_3 && -\sigma_3(y_3 + \sigma_2(y_2 + \sigma_1(y_1)))\\
\dot{y}_3 &= && -\sigma_3(y_3 + \sigma_2(y_2 + \sigma_1(y_1)))
\end{aligned}$$

How does y_3 evolve ?
Let $V_3 = y_3^2 \Rightarrow$

$$\dot{V}_3 = -2y_3\sigma_3(y_3 + \sigma_2(y_2 + \sigma_1(y_1)))$$

As $|\sigma_2(.)| \leq M_2 < \frac{1}{2}L_3$,
$\dot{V}_3 < 0$ for all $|y_3| > \frac{1}{2}L_3$

$\Rightarrow |y_3|$ will decrease.

In finite time $|y_3|$ will be $< \frac{1}{2}L_3$ and σ_3 will now operate in the linear region.
(Note: no finite escape for the other states.)

$$\begin{aligned}
\dot{y}_2 &= y_3 - (y_3 + \sigma_2(y_2 + \sigma_1(y_1)))\\
&= -\sigma_2(y_2 + \sigma_1(y_1)))
\end{aligned}$$

Same kind of argument shows us that after finite time, the closed loop will look like

$$\begin{aligned}
\dot{y}_1 &= -y_1\\
\dot{y}_2 &= -y_1 - y_2\\
\dot{y}_3 &= -y_1 - y_2 - y_3
\end{aligned}$$

i.e. after a finite time, the dynamics are exponentially stable

Remark:
Although we have found a globally stabilizing, bounded, control law, u, the internal states may have huge overshoots !!

Integrator forwarding

strict-feedforward systems

$$\begin{aligned}
\dot{x}_1 &= x_2 + f_1(x_2, x_3, \dots, x_n, u)\\
&\vdots\\
\dot{x}_{n-1} &= x_n + f_{n-1}(x_n, u)\\
\dot{x}_n &= u
\end{aligned}$$

Due to the lack of feedback connections, solutions always exists and are of the form

$$\begin{aligned}
x_n(t) &= x_n(0) + \int_0^t u(s)ds\\
x_{n-1}(t) &= x_{n-1}(0) + \int_0^t (x_n(s) + f_{n-1}(x_n(s), u(s)))ds\\
&\vdots
\end{aligned}$$

1. Begin with stabilizing the system $\dot{x}_n = u_n$
   Use e.g. $V_n = x_n^2$ and $u_n = -x_n$
2. Augment the control law
   $u_{n-1}(x_{n-1}, x_n) = u_n(x_n) + v_{n-1}$
   such that u_{n-1} stabilizes the cascade

$$\begin{aligned} \dot{x}_{n-1} &= x_n + f_{n-1}(x_n, u) \\ \dot{x}_n &= u_{n-1} \end{aligned}$$

...

k. Augment the control law
   $u_k(x_k, x_{k+1}) = u_n(x_{k+1}) + v_k$
   such that u_k stabilizes the cascade

$$\begin{aligned} \dot{x}_k &= x_{k+1} + f_k(\dots) \\ \dot{X}_{k+1} &= F_{k+1}(\dots, u_k) \end{aligned}$$

How is the cascade (in step k) stabilized?

We have a cascade of one GAS/LES system and a ISS-system with a linear growth-condition.

There exists a Lyapunov function for the (sub-) system

$$V_k = V_{k+1} + \frac{1}{2}x_k^2 + \int_0^\infty x_k(s) f_k(X_{k+1}(s))ds$$

It can be shown that $\dot{V}_k|_{u_k = -L_g V_k} < 0$ and finally u_1 minimizes a cost functional of the form

$$J = \int_0^\infty (l(x) + u^2)ds$$

The cross-term can only be exactly evaluated for very simple systems. In other cases it has to be numerically evaluated or approximated by i.e. Taylor series

Connection to Teel's results:
To avoid computations of the integrals we can use nested low-gain (saturated) control.
Also showed to be GAS/LES for the integrator chain, but LAS/LES for the general strict-feedforward system.

(Compare with high-gain design in backstepping)

Can use a feedback passivation design for a system if

1. A relative degree condition satisfied
2. The system is weakly minimum phase

Backstepping is a recursive way of finding a relative degree one output.

Integrator forwarding allows us to stabilize weakly non-minimum phase systems.

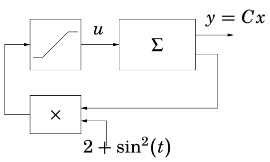
Conclusions

- Global/semiglobal stabilization of strict-feedforward system
  (No exact linearization possible)
- Tracking results reported
- Relaxes weakly minimum phase-condition
- Integration forwarding - "necessary" to simplify controller

Motivation: Simple example

Consider the following simple feedback system

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} u = Ax + Bu \qquad (\Sigma) \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} x = Cx \\ u = sat(x_2 \cdot (2 + \sin^2(t))) \end{cases}$$



Example cont'd

- linear subsystem unstable
- input saturation $\Rightarrow$ *At best local stability.*

——————— $Tools$ ———————

Locally valid Quadratic Contraint (QC) $(sector\ condition)$

$$0 \leq (\kappa_2 \cdot x_2 - u)(u - \kappa_1 \cdot x_2) =$$

$$\begin{bmatrix} x_1 & x_2 & u \end{bmatrix} \begin{bmatrix} \begin{pmatrix} 0 & 0 \\ 0 & -3 \\ (0 & 2) \end{pmatrix} & \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ u \end{bmatrix} \text{ for some } |x_2| < c$$

$$\begin{aligned} \kappa_1 &= 1 && \text{Lower bound :} \\ & && \text{'linear feedback stability cond.'} \\ & && u = \kappa x_2,\ \kappa \in (1, \infty) \\ \kappa_2 &= 3 && \text{Upper bound :} \\ & && \text{sector of nonlinearity} \end{aligned}$$

Preliminaries

State feedback

$$\begin{cases} \dot{x} = Ax + Bu = Ax + B\phi(x) \\ y = Cx \\ u = \phi(x) \end{cases}$$

Observer feedback

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \\ \dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}) \\ u = \phi(\hat{x}) \end{cases}$$

Asymptotically stable for state feedback $u = \phi(x)$

Re-write with *error dynamics* ($e = \hat{x} - x$)

$$\begin{cases} \dot{e} = (A - LC)e \\ \dot{x} = Ax + B\phi(x + e) + LCe \\ u = \phi(\hat{x}) \end{cases}$$