LionSealGrey

Optimal Control 2018

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L1: Functional minimization, Calculus of variations (CV) problem

L2: Constrained CV problems, From CV to optimal control

L3: Maximum principle, Existence of optimal control

L4: **Maximum principle (proof)**

L5: Dynamic programming, Hamilton-Jacobi-Bellman equation

L6: Linear quadratic regulator

L7: Numerical methods for optimal control problems

Exercise sessions (20%):
Solve 50% of problems in advance. Hand-in later.
Mini-project (20%):
Study and present your own optimal control problem.
Written take-home exam (60%).

Summary of L3: Basic problem formulation

Find a control $u \in U \subset \mathbb{R}^m$ that minimizes the cost

$$J(u) = \int_{t_0}^{t_f} \underbrace{L(x(t), u(t))}_{\text{time independent}} dt + K(x_f)$$

where

- $\dot{x} = \underbrace{f(x(t), u(t))}_{\text{time independent}},\ x(t_0) = x_0,\ x \in \mathbb{R}^n, K(x_f) = 0, (t_f, x_f) \in S$
- f, f_x, L, L_x continuous

  - **Basic fixed-endpoint problem (BFEP)** (t_f free, x_f fixed)

    $S = [t_0, \infty) \times \{x_1\}$
  - **Basic variable-endpoint problem (BVEP)** (t_f free, $x_f \in S_1$)

    $S = [t_0, \infty) \times S_1$

    $S_1 = \{x \in \mathbb{R}^n : h_1(x) = h_2(x) = \cdots h_{n-k}(x) = 0\}$

    $h_i \in \mathcal{C}^1(\mathbb{R}^n \to \mathbb{R}), i = 1, \ldots, n-k.$

Summary of L3: Maximum principle

Define the Hamiltonian

$$H(x, u, p, p_0) = \langle p, f(x, u) \rangle + p_0 L(x, u).$$

Assume that the basic problem has a solution $(u^*(t), x^*(t))$. Then there exist a function $p^* : [t_0, t_f] \to \mathbb{R}^n$ and a constant $p_0^* \leq 0$ satisfying $(p_0^*, p^*(t)) \neq (0, 0)$ $\forall t \in [t_0, t_f]$ and

$$1)\ \dot{x}^* = H_p(t, x^*, u^*, p^*),\ \dot{p}^* = -H_x(t, x^*, u^*, p^*).$$

$$2)\ H(x^*(t), u^*(t), p^*(t), p_0^*) \geq H(x^*(t), u(t), p^*(t), p_0^*)$$
$$\forall t \in [t_0, t_f],\ \forall u \in U.$$

$$3)\ H(x^*(t), u^*(t), p^*(t), p_0^*) = 0 \quad \forall t \in [t_0, t_f]$$

$4)\ \langle p^*(t_f), d \rangle = 0 \quad \forall d \in T_{x^*(t_f)} S_1$ **(Only for BVEP)**

$T_{x^*(t_f)} S_1$: tangent space to S_1. *Transversality condition.*

Summary of L3: Transversality condition

$$\langle p^*(t_f), d\rangle = 0 \quad \forall d \in T_{x^*(t_f)} S_1. \tag{1}$$

$$T_{x^*(t_f)} S_1 = \{d \in \mathbb{R}^n : \langle \nabla h_i(x^*(t_f)), d\rangle = 0, i = 1, \dots n-k\}$$

- (1) means $p^*(t_f)$ is a linear combination of $\nabla h_i(x^*(t_f))$.
- $S_1 = \{x_1\} \implies$ (1) is true for all $p^*(t_f)$.
- $S_1 = \mathbb{R}^n$ (i.e., $k = n$) $\implies p^*(t_f) = 0$.
- In general, k degrees of freedom for $x^*(t_f)$ and $n-k$ degrees of freedom for $p^*(t_f)$.

Outline

- **Proof of Maximum Principle** consists of several steps:
  - S1: From Lagrange form to Mayer form
  - S2: Temporal control perturbation
  - S3: Spatial control perturbation
  - S4: Variational equation
  - S5: Terminal cone
  - S6: Key topological lemma
  - S7: Separating hyperplane
  - S8: Adjoint equation
  - S9: Hamiltonian properties
  - S10: Transversality condition

1st Step of the Proof

S1: **From Lagrange to Mayer form**

An auxiliary state variable

$$x^0(t) = \int_{t_0}^{t} L(x(\tau), u(\tau))d\tau,\ x^0(t_0)$$

results in the augmented system

$$\dot{x}^0 = L(x, u),\ x^0(t_0) = 0$$

$$\dot{x} = f(x, u),\ x(t_0) = x_0$$

with the cost

$$J(u) = \int_{t_0}^{t_f} L(x(t), u(t))dt = x^0(t_f)$$

1st Step of the Proof (continued)

System representation

$$\dot{y} = \begin{pmatrix} L(x,u) \\ f(x,u) \end{pmatrix} =: g(x,u)$$

in terms of

$$y = \begin{pmatrix} x^0 \\ x \end{pmatrix} \in \mathbb{R}^{n+1}$$

results in the Mayer problem

$$J(u) = x^0(t_f) \to \min$$

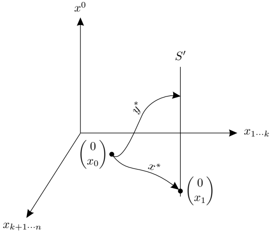

Figure 4.1: The optimal trajectory of the augmented system

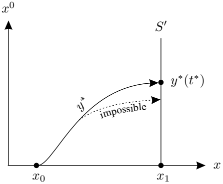

Figure 4.2: Principle of optimality

2nd Step of the Proof

S2: **Temporal control perturbation**

Control variation at the terminal time instant

$$u_\tau(t) := u^*(\min\{t, t^*\}), \quad t \in [t_0, t + \varepsilon\tau]$$

with an arbitrary $\tau \in \mathbb{R}$ and a small $\varepsilon > 0$, and a new terminal time $t^* + \varepsilon\tau$.

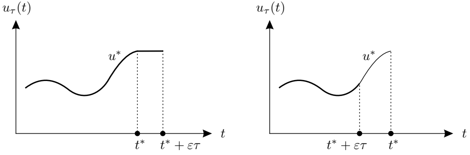

Figure 4.3: A temporal perturbation

2nd Step of the Proof (continued)

Taylor expansion around $t = t^*$

$$y(t^* + \varepsilon\tau) = y^*(t^*) + \dot{y}(t^*)\varepsilon\tau + o(\tau) = y^*(t^*) + g(y^*(t^*), u^*(t^*))\varepsilon\tau + o(\tau) = y^*(t^*) + \varepsilon\delta(\tau) + o(\tau)$$

determines the **trajectory variation** $\varepsilon\delta(\tau)$ at the terminal point.

Varying $\tau \in \mathbb{R}$ under fixed ε forms a direction $\bar{\rho}$ at $y^*(t^*)$ of $y^*(t^*) + \varepsilon\delta(\tau)$.

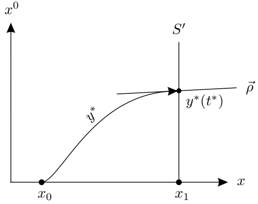

Figure 4.4: The effect of a temporal control perturbation

3rd Step of the Proof

S3: **Spatial control perturbation**

Needle control variation

$$u_{\omega,I}(t) = \begin{cases} \omega & \text{if } t \in I \\ u^*(t) & \text{otherwise} \end{cases}$$

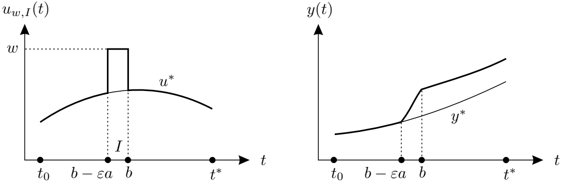

Figure 4.5: A spatial control perturbation and its effect on the trajectory

3rd Step of the Proof (continued)

Throughout, **symbol** $\approx$ stands for equality up to terms of $o(\varepsilon)$:

Taylor expansion $\quad y^*(b-\varepsilon a) \approx y^*(b) - \dot{y}^*(b)\varepsilon a \quad$ at $\quad t = b$

Due to the state equation, it follows

$$y^*(b) \approx y^*(b-\varepsilon a) + g(y^*(b), u^*(b))\varepsilon a$$

On the other hand

Taylor expansion $\quad y(b) \approx y(b-\varepsilon a) + \dot{y}(b-\varepsilon a)\varepsilon a \quad$ at $\quad t = b - \varepsilon a$

$$y(b-\varepsilon a) = y^*(b-\varepsilon a) \quad \Downarrow \quad \dot{y}(t) = g(y, u_{\omega,I})$$

$$y(b) \approx y^*(b-\varepsilon a) + g(y^*(b-\varepsilon a), \omega)\varepsilon a$$

Moreover, $y(b) \approx y^*(b-\varepsilon a) + g(y^*(b), \omega)\varepsilon a$ because

3rd Step of the Proof (continued)

Taylor expansion

$$g(y^*(b-\varepsilon a),\omega)\varepsilon a \approx g(y^*(b),\omega)\varepsilon a + g_y(y^*(b),\omega)[y^*(b-\varepsilon a)-y^*(b)]\varepsilon a$$

captures the second term of the order ε^2.

Comparing

$$\begin{aligned} y^*(b) &\approx y^*(b-\varepsilon a) + g(y^*(b),u^*(b))\varepsilon a \\ y(b) &\approx y^*(b-\varepsilon a) + g(y^*(b),\omega)\varepsilon a \end{aligned}$$

one concludes $y(b) \approx y^*(b) + \nu_b(\omega)\varepsilon a$ where

$$\nu_b(\omega) = g(y^*(b),\omega) - g(y^*(b),u^*(b))$$

4th Step of the Proof

S4: **Variational equation**

The current goal is to study the propagation $\psi(t) : [b, t^*] \to \mathbb{R}^{n+1}$ of the deviation of the perturbed trajectory from the optimal one:

$$y(t) = y^*(t) + \varepsilon\psi(t) + o(\varepsilon) =: y(t, \varepsilon)$$

where it was just shown that $\psi(b) = \nu_b(\omega)a$. Also it is clear that

$$\psi(t) = y_\varepsilon(t, 0)$$

The perturbed trajectory is governed by the integral equation

$$y(t, \varepsilon) = y(b, \varepsilon) + \int_b^t g(y(s, \varepsilon), u^*(s))ds$$

4th Step of the Proof (continued)

Differentiating the integral equation at $\varepsilon = 0$ yields

$$\underbrace{y_\varepsilon(t,0)}_{\psi(t)} = \nu_b(\omega)a + \int_b^t g_y(y(s,0),u^*(s))\,\underbrace{y_\varepsilon(s,0)}_{\psi(s)}\,ds$$

thereby resulting in

$$\psi(t) = \nu_b(\omega)a + \int_b^t g_y(y^*(s),u^*(s))\psi(s)ds$$

It follows

Variational equation $\quad \dot\psi = g_y(y^*,u^*) = \underbrace{g_y\big|_*}_{A_*(t)}\psi = A_*(t)\psi$

or in terms of $\psi = (\eta^0, \eta^T)^T$:

$$\begin{aligned} \dot\eta^0 &= (L_x)^T\big|_*\eta, \\ \dot\eta &= f_x\big|_*\eta \end{aligned}$$

4th Step of the Proof (continued)

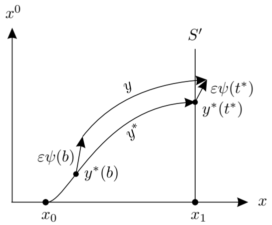

Figure 4.6: Propagation of a spatial perturbation

Summarizing

$$y(t^*) = y^*(t^*) + \varepsilon\psi(t^*) + o(\varepsilon)$$

where

$$\psi(t^*) = \Phi_*(t^*, b)\psi(b) = \Phi_*(t^*, b)\nu_b(\omega)a$$

provided that $\Phi_*(\cdot, \cdot)$ is the state transition matrix for $\dot{\psi} = A_*(t)\psi$.

5th Step of the Proof

S5: **Terminal cone**

Resulting state variation $\quad y(t^*) = y^*(t^*) + \varepsilon\Phi_*(t^*, b)\nu_b(\omega)a + o(\varepsilon)$

Setting

$$\delta(\omega, I) := \Phi_*(t^*, b)\nu_b(\omega)a$$

yields

$$y(t^*) = y^*(t^*) + \varepsilon\delta(\omega, I) + o(\varepsilon)$$

where the direction $\bar{\rho}(\omega, b)$ of $\delta(\omega, I)$ does not depend of the scalar a.

All admissible rays $\bar{\rho}(\omega, b)$ form a cone $\bar{P}$ with vertex at $y^*(t^*)$.

In general, $\bar{P}$ is non-convex.

5th Step of the Proof (continued)

Question: Is there a perturbation, resulting in $\varepsilon\delta(\omega_1, I_1) + \varepsilon\delta(\omega_2, I_2)$ for some control values ω_1, ω_2 and intervals $I_1 = (b_1 - \varepsilon a_1, b_1)$, $I_2 = (b_2 - \varepsilon a_2, b_2)$, and ε small enough?

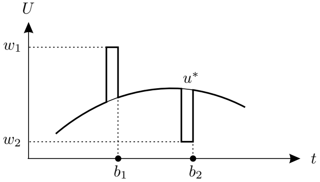

Figure 4.7: “Adding” spatial perturbations

Answer: Yes because of the linearity of the variational equation.

5th Step of the Proof (continued)

Indeed, as has been shown

$$y(b_1) = y^*(b_1) + \underbrace{\nu_{b_1}(\omega_1)\varepsilon a_1}_{\varepsilon\psi(b_1)} + o(\varepsilon)$$

at the end of the first perturbation interval. Then

$$y(b_2) = y^*(b_2) + \varepsilon\big[\Phi_*(b_2, b_1)\nu_{b_1}(\omega_1)a_1 + \nu_{b_2}(\omega_2)a_2\big] + o(\varepsilon)$$

at the end of the second perturbation interval.

Finally, by the semigroup property $\Phi_*(t^*, b_2)\Phi_*(b_2, b_1) = \Phi_*(t^*, b_1)$,

$$\begin{aligned}
y(t^*) &= y^*(t^*) + \varepsilon\Phi_*(t^*, b_2)\big[\Phi_*(b_2, b_1)\nu_{b_1}(\omega_1)a_1 + \nu_{b_2}(\omega_2)a_2\big] + o(\varepsilon) \\
&= y^*(t^*) + \varepsilon\Phi_*(t^*, b_1)\nu_{b_1}(\omega_1)a_1 + \varepsilon\Phi_*(t^*, b_2)\nu_{b_2}(\omega_2)a_2 + o(\varepsilon) \\
&= y^*(t^*) + \varepsilon\delta(\omega_1, I_1) + \varepsilon\delta(\omega_1, I_1) + o(\varepsilon).
\end{aligned}$$

5th Step of the Proof (continued)

The terminal cone C_{t^*} is the set of points of the form

$$y = y(t^*) + \varepsilon\left[\beta_0 + \Sigma_{i=1}^{m}\beta_1\delta(\omega_i, I_i)\right]$$

where $\varepsilon > 0$, $\beta_0, \beta_1, \ldots, \beta_m \geq 0$, the temporal variation $\delta(\tau)$ comes with some τ, and the spatial variations $\delta(\omega_i, I_i)$ come with some ω_i and I_i.

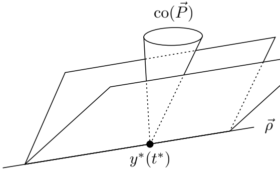

Figure 4.8: The terminal cone

The principal feature: $\forall\, y \in C_{t^*}$ $\exists$ a perturbation of u^* such that the terminal point $y(t_f)$ satisfies $y(t_f) = y + o(\varepsilon)$

(follows from the linearity of the variational equation and the linear dependence of $\delta(\tau)$ on τ)

6th Step of the Proof

S6: Key topological lemma

The optimality of u^* is now in play.

Let $\bar{\mu}$ be the ray, originated at $y^*(t^*)$ and generated by the downward-pointed vector

$$\mu := (-1, 0 \cdots 0)^T \in \mathbb{R}^{n+1}$$

Due to the optimality, $\bar{\mu}$ is to be directed outside of the cone C_{t^*}

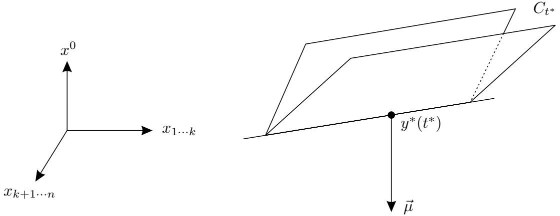

Figure 4.9: Illustrating the statement of Lemma 4.1

Lemma

$\bar{\mu}$ does not intersect the interior of C_{t^*}.

6th Step of the Proof (continued)

Suppose **Lemma is false.** Then $\exists \hat{y} \in \bar{\mu}$ below $y^*(t^*)$ such that $\hat{y} \in C_{t^*}$ together with a ball $B_\varepsilon \subset C_{t^*}$ $\Rightarrow$ For a suitable $\beta > 0$, one has

$$\hat{y} = y^*(t^*) + \varepsilon\beta\mu$$

Since $B_\varepsilon \subset C_{t^*}$, its points are of the form $y^*(t^*) + \varepsilon\nu$ where $\varepsilon\nu$ are first-order perturbations, arising from the earlier control perturbations.

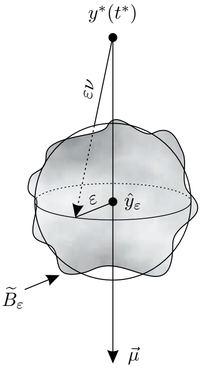

Figure 4.10: Proving Lemma 4.1

- Actual terminal points $y^*(t^*) + \varepsilon\nu + o(\varepsilon)$ of these perturbed trajectories form the set $\tilde{B}_\varepsilon$ which is $o(\varepsilon)$ away from B_ε
- Let $\varepsilon \to 0$, then $\hat{y} := y^*((t^*) + \varepsilon\beta\mu$ approaches $y^*((t^*)$.
- Since the center of B_ε is on $\hat{\mu}$ below $y^*(t^*)$ then for sufficiently small ε, set $\tilde{B}_\varepsilon$ intersects $\bar{\mu}$ below $y^*(t^*)$, too **that contradicts the optimality.**

7th Step of the Proof

S7: **Separating hyperplane**

Theorem (Separating Hyperplane Theorem of Convex Analysis)

There exists a hyperplane separating two nonempty disjoint convex sets.

By Theorem, there exists a plane, separating the ray $\bar{\mu}$ from the interior[1] of C_{t^*}, and hence from C_{t^*} itself.

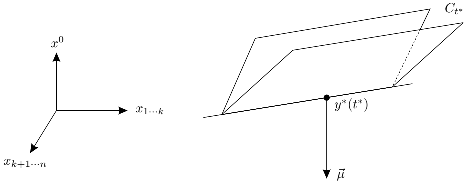

Figure 4.9: Illustrating the statement of Lemma 4.1

[1] If the interior of C_{t^*} is empty then there exists a plane that contains C_{t^*}, thus separating trivially C_{t^*} and $\bar{\mu}$.

7th Step of the Proof (continued)

Let

$$P^* = \begin{pmatrix} p_0^* \\ p^*(t^*) \end{pmatrix} \in \mathbb{R}^{n+1}$$

be the normal vector to the separating hyperplane.

Hyperplane equation: $< P^*, y >=< P^*, y^*(t^*) >$

Separation property is analytically formalized as

$$< P^*, \delta > \leq 0 \quad \forall \delta : y^*(t^*) + \delta \in C_{t^*}$$

and

$$< P^*, \mu > \geq 0$$

where $\mu = (-1, 0, \ldots, 0)^T$ is the generator of the ray $\bar{\mu}$.

The latter property requires $p_0^* \leq 0$ whereas the former property serves as the to-be-defined terminal condition for the adjoint system.

8th Step of the Proof

S8: **Adjoint equation**

Linear time-varying systems, governed by

$$\dot{x} = Ax, \quad \dot{z} = -A^T z,$$

are adjoint to each other.

The inner product of their solutions remains constant:

$$\frac{d}{dt} < z, x > = < \dot{z}, x > + < z, \dot{x} > = (-A^T z)^T x + z^T A x = 0$$

Variational equations $\quad \dot{\eta}^0 = (L_x)^T\big|_* \eta, \quad \dot{\eta} = f_x\big|_* \eta$

Adjoint equations $\quad \dot{p}_0 = 0, \quad \dot{p} = -(L_x)\big|_* p_0 - (f_x)^T\big|_* p$

8th Step of the Proof (continued)

Coupling the adjoint equations to the terminal conditions determined by the separating hyperplane, yields $p_0(t) = p_0^* \leq 0$ while the latter equation is represented in the canonical Hamiltonian form

$$\dot{p} = -H_x(x^*, u^*, p, p_0^*)$$

thus establishing the **first statement of the maximum principle.**

By the property of the inner product to remain constant for the adjoint variable $P^*(t) = \left(p_0^*(t), p^*(t)^T\right)^T$, one concludes

$$< P^*(t), \psi(t) > = < P^*(t^*), \psi(t^*) > \quad \forall t \in [t_0, t^*]$$

for any solution $\psi = (\eta^0, \eta^T)^T$ of the variational equation.

Since the normal vector $\begin{pmatrix} p_0^* \\ p^*(t^*) \end{pmatrix}$ to the separating hyperplane is nontrivial, the solution of the LTV adjoint system remains nonzero

$$\begin{pmatrix} p_0^* \\ p^*(t) \end{pmatrix} \neq 0 \quad \forall t \in [t_0, t^*]$$

as required by the **maximum principle.**

9th Step of the Proof

S9a): **Hamiltonian maximization condition**

Infinitesimal state variation of the terminal point

$$y(t^*) \approx y^*(t^*) + \varepsilon \Phi_*(t^*, b)\nu_b(\omega)a \in C_{t^*}$$

Thus, taking into account $a > 0$ and $\varepsilon > 0$, and applying the separating hyperplane property

$$< P^*, \delta > \leq 0 \quad \forall \delta : y^*(t^*) + \delta \in C_{t^*}$$

to $\delta = \varepsilon \Phi_*(t^*, b)\nu_b(\omega)a$ yield

$$< P^*(t^*), \Phi_*(t^*, b)\nu_b(\omega) > \leq 0$$

where the adjoint variable $P^*(t) = \left(p_0^*(t), p^*(t)^T\right)^T$

Hamiltonian maximization condition (continued)

By invoking the adjoint inner property

$$< P^*(t), \psi(t) > = < P^*(t^*), \psi(t^*) > \quad \forall t \in [t_0, t^*]$$

for the variational equation solution $\psi(t) := \Phi_*(t^*, b)\nu_b(\omega)$, initialized with $\psi(b) = \nu_b(\omega)$, it follows that $< P^*(b), \nu_b(\omega) > \leq 0.$ Since

$$\nu_b(\omega) = g(y^*(b), \omega) - g(y^*(b), u^*(b)) = \begin{pmatrix} L(x^*(b), \omega) - L(x^*(b), u^*b)) \\ f(x^*(b), \omega) - f(x^*(b), u^*b)) \end{pmatrix}$$

and $P^*(b) = \begin{pmatrix} p_0^* \\ p^*(b) \end{pmatrix}$, it follows

$$\underbrace{\left\langle \begin{pmatrix} p_0^* \\ p^*(b) \end{pmatrix}, \begin{pmatrix} L(x^*(b), \omega) \\ f(x^*(b), \omega) \end{pmatrix} \right\rangle}_{H(x^*(b), \omega, p^*(b), p_0^*)} \leq \underbrace{\left\langle \begin{pmatrix} p_0^* \\ p^*(b) \end{pmatrix}, \begin{pmatrix} L(x^*(b), u^*(b) \\ f(x^*(b), u^*(b)) \end{pmatrix} \right\rangle}_{H(x^*(b), u^*(b), p^*(b), p_0^*)}$$

9th Step of the Proof

S9b): $H|_* \equiv 0$

The separation property

$$< P^*, \delta > \leq 0 \quad \forall \delta : y^*(t^*) + \delta \in C_{t^*}$$

applies,in particular, to

$$\delta(\tau) = \begin{pmatrix} L(x^*(t^*), u^*(t^*) \\ f(x^*(t^*), u^*(t^*)) \end{pmatrix} \tau \in C_{t^*}$$

Since τ can either be positive or negative it follows

$$\underbrace{\left\langle \begin{pmatrix} p_0^* \\ p^*(t^*) \end{pmatrix}, \begin{pmatrix} L(x^*(t^*), u^*(t^*) \\ f(x^*(t^*), u^*(t^*)) \end{pmatrix} \right\rangle}_{H(x^*(t^*), u^*(t^*), p^*(t^*), p_0^*)} = 0$$

9th Step of the Proof (continued)

$H|_*(\cdot) = H(x^*(\cdot), u^*(\cdot), p^*(\cdot), p_0^*)$ is continuous in time.

Indeed, by the Hamiltonian maximization property:

$$\underbrace{\lim_{b \to t^-} H(x^*(b), \overbrace{\omega}^{u^*(t^+)}, p^*(b), p_0^*)}_{H(x^*(t), u^*(t^+), p^*(t), p_0^*)} \leq \underbrace{\lim_{b \to t^-} H(x^*(b), u^*(b), p^*(t), p_0^*)}_{H(x^*(t), u^*(t^-), p^*(t), p_0^*)} \tag{2}$$

$$\underbrace{\lim_{b \to t^+} H(x^*(b), u^*(b), p^*(b), p_0^*)}_{H(x^*(t), u^*(t^*+), p^*(t), p_0^*)} \geq \underbrace{\lim_{b \to t^+} H(x^*(b), \overbrace{\omega}^{u^*(t^-)}, p^*(t), p_0^*)}_{H(x^*(t), u^*(t^-), p^*(t), p_0^*)} \tag{3}$$

It follows $H(x^*(t), u^*(t^-), p^*(t), p_0^*) = H(x^*(t), u^*(t^+), p^*(t), p_0^*)$.

9th Step of the Proof (continued)

Thus, $H|_*(\cdot)$ is continuous and $H|_*(t^*) = 0 \Rightarrow$ it remains to show

$$\dot{H}|_*(\cdot) = 0 \quad \text{a.e.} \tag{4}$$

Function $m(x,p) := \max_{u \in U} H(x,u,p,p_0^*)$ is Lipschitz (hence, absolutely) continuous in time along $x = x^*(t), p = p^*(t)$ because of

$$\begin{aligned} H(x^*(t'), u^*(t), p^*(t'), p_0^*) - H(x^*(t), u^*(t), p^*(t), p_0^*) \leq \\ m(x^*(t'), p^*(t')) - m(x^*(t), p^*(t), p_0^* \leq \\ H(x^*(t'), u^*(t'), p^*(t'), p_0^*) - H(x^*(t), u^*(t'), p^*(t), p_0^*) \end{aligned}$$

and assumptions on the system in question. Thus, by Liberzon's Exersise 4.6 your homework, the Hamiltonian property (4) is concluded.

10th Step of the Proof

S10: **Transversality condition for BVEP** $(x(t_f) \in S_1)$

Set $D \subset \mathbb{R}^{n+1}$ of all $y = \begin{pmatrix} x^0 \\ x \end{pmatrix} \in \mathbb{R}^{n+1}$ with x^0-component lower then the optimal cost, whose x-component is in S_1. Its linear approximation is the linear span of $\bar{\mu}$ and the tangent space $T_{x^*(t^*)}S_1$:

$$T := \left\{ y \in \mathbb{R}^{n+1} : y = y^*(t^*) + \begin{pmatrix} 0 \\ d \end{pmatrix} + \beta\mu,\ d \in T_{x^*(t^*)}S_1,\ \beta \geq 0 \right\}$$

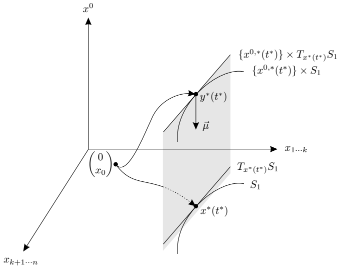

Figure 4.12: Illustrating the construction of the set T

10th Step of the Proof (continued)

Lemma

T does not intersect the interior of the cone C_{t^*}.

Proof is similar to that of Lemma 4.1.

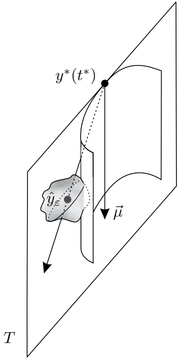

Figure 4.13: Proving Lemma 4.2

By Lemma 4.2 and Separating Hyperplane Theorem, there exists a hyperplane that separates T and C_{t^*}.

10th Step of the Proof (continued)

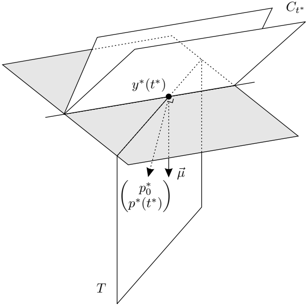

Figure 4.14: A separating hyperplane for the Basic Variable-Endpoint Control Problem

Writing the separation property for vectors in T with $\beta = 0$ yields

$$\left\langle \begin{pmatrix} p_0^* \\ p^*(t^*) \end{pmatrix}, \begin{pmatrix} 0 \\ d \end{pmatrix} \right\rangle = \langle p^*(t^*), d \rangle \geq 0 \quad \forall d \in T_{x^*(t^*)} S_1$$

where p_0^* and $p^*(t^*))$ are the components of the normal vector to the separating hyperplane.

10th Step of the Proof (continued)

Since $d \in T_{x^*(t^*)} S_1 \Rightarrow -d \in T_{x^*(t^*)} S_1$ it follows

$$\langle p^*(t_f), d \rangle = 0 \quad \forall d \in T_{x^*(t_f)} S_1$$

where

$$T_{x^*(t_f)} S_1 = \{ d \in \mathbb{R}^n : \langle \nabla h_i(x^*(t_f)), d \rangle = 0, i = 1, \ldots n - k \}$$

The BVEP transversality condition is thus established.

The proof is completed!