Circuit Theory

Richard Pates

Who cares?

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–Malcolm Smith

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...turned into the major scientific interest of Kalman's last decade...

–Malcolm Smith

Who cares?

Who cares?

- A very useful class of models:
  - Can describe many phenomena
  - Many analogues with other physical domains
  - Many useful control architectures (PID, phase lead/lag)
- Inspires useful theory:
  - Lyapunov functions
  - Energy dissipation and passivity
  - ...

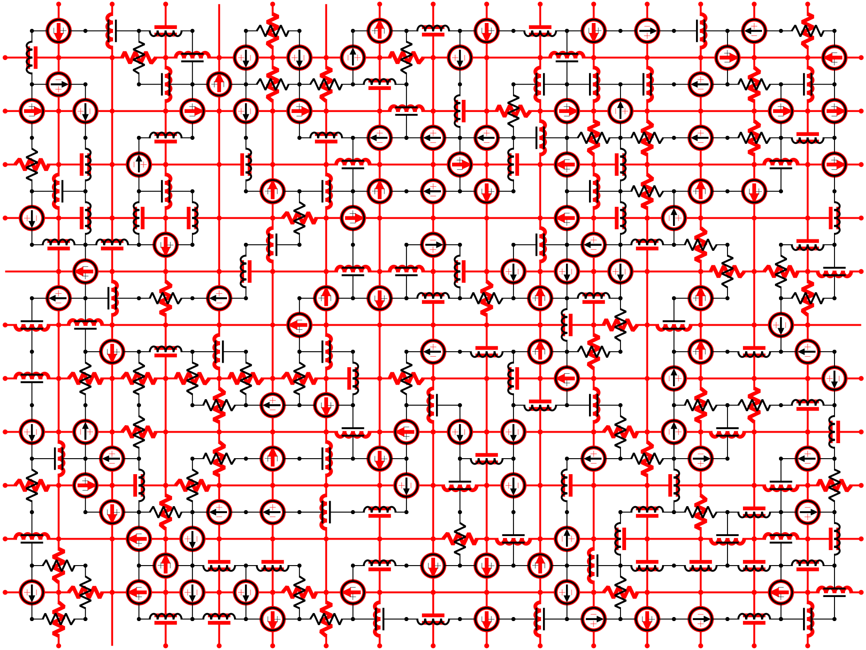
Today's Lecture

- Modelling electrical networks
- Analogues
- Network synthesis and state-space models
- Unsolved problems

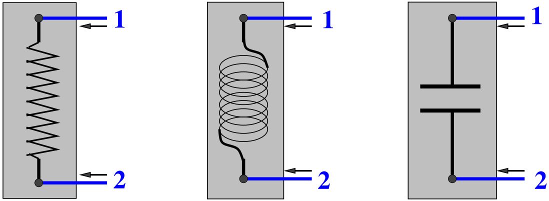
Modelling circuits

A picture, with a clear mathematical meaning:

- edges $\iff$ differential equations, driving points
- topology $\iff$ algebraic equations

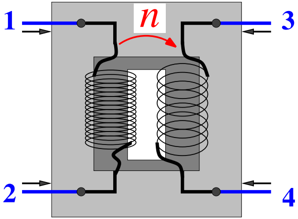


The passive elements



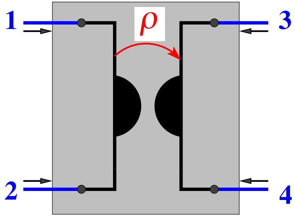
$$\left[\; P_k\left(\tfrac{d}{dt}\right) \;\middle|\; Q_k\left(\tfrac{d}{dt}\right) \;\right] \left[\frac{i_{\text{int},k}}{v_{\text{int},k}}\right] = 0$$

The passive elements



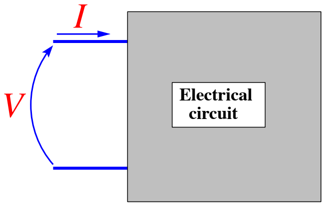
$$\left[\begin{array}{cc|cc} 0 & 0 & n & -1 \\ 1 & n & 0 & 0 \end{array}\right] \left[\begin{array}{c} i_{\text{int},j} \\ i_{\text{int},k} \\ \hline v_{\text{int},j} \\ v_{\text{int},k} \end{array}\right] = 0$$

The passive elements



$$\left[\begin{array}{cc|cc} 0 & -\rho & 1 & 0 \\ \rho & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c} i_{\text{int},j} \\ i_{\text{int},k} \\ \hline v_{\text{int},j} \\ v_{\text{int},k} \end{array}\right] = 0$$

Driving points



Pair of terminals with *external* through current and across voltage

Algebraic equations

$$\left[\begin{array}{cc|cc} M_1 & M_2 & M_3 & M_4 \end{array}\right] \left[\begin{array}{c} i_{\text{int}} \\ v_{\text{int}} \\ \hline i_{\text{ext}} \\ v_{\text{ext}} \end{array}\right] = 0$$

Circuit behavior

Element laws, driving points, conservation laws:

$$\left[\begin{array}{cc|cc} M_1 & M_2 & M_3 & M_4 \\ \hline \operatorname{diag} P_k\left(\frac{d}{dt}\right) & \operatorname{diag} Q_k\left(\frac{d}{dt}\right) & 0 & 0 \end{array}\right] \left[\begin{array}{c} i_{\text{int}} \\ v_{\text{int}} \\ \hline i_{\text{ext}} \\ v_{\text{ext}} \end{array}\right] = 0 \tag{1}$$

Circuit behavior

Element laws, driving points, conservation laws:

$$\left[\begin{array}{cc|cc} M_1 & M_2 & M_3 & M_4 \\ \hline \operatorname{diag} P_k\left(\frac{d}{dt}\right) & \operatorname{diag} Q_k\left(\frac{d}{dt}\right) & 0 & 0 \end{array}\right] \left[\begin{array}{c} i_{\text{int}} \\ v_{\text{int}} \\ \hline i_{\text{ext}} \\ v_{\text{ext}} \end{array}\right] = 0 \qquad (1)$$

Full behavior

$$\mathscr{B} = \{(i,v) : (i,v) \in \mathscr{L}_{\text{loc}} \times \mathscr{L}_{\text{loc}} \text{ satisfy (1)}\}$$

Circuit behavior

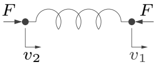
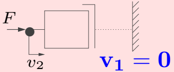
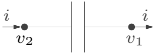
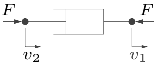
Element laws, driving points, conservation laws:

$$\left[\begin{array}{cc|cc} M_1 & M_2 & M_3 & M_4 \\ \hline \operatorname{diag} P_k\left(\frac{d}{dt}\right) & \operatorname{diag} Q_k\left(\frac{d}{dt}\right) & 0 & 0 \end{array}\right] \left[\begin{array}{c} i_{\text{int}} \\ v_{\text{int}} \\ \hline i_{\text{ext}} \\ v_{\text{ext}} \end{array}\right] = 0 \qquad (1)$$

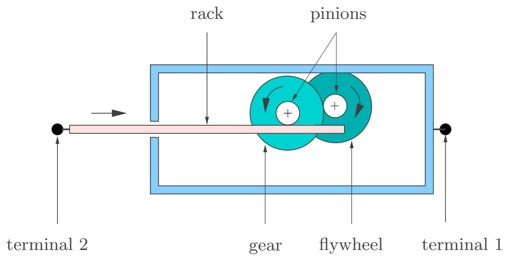
External behavior

$$\mathscr{B} = \left\{ (i_e, v_e) : \left(\begin{bmatrix} i_i \\ i_e \end{bmatrix}, \begin{bmatrix} v_i \\ v_e \end{bmatrix} \right) \in \mathscr{L}_{\text{loc}} \times \mathscr{L}_{\text{loc}} \text{ satisfy (1)} \right\}$$

Analogues

Mechanical	Electrical
 spring	 inductor
 mass	 capacitor
 damper	 resistor

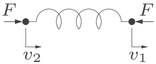
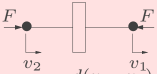
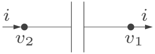
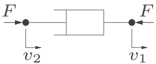
Analogues



Suppose the flywheel of mass m rotates by α radians per meter of relative displacement between the terminals. Then:

$$\mathbf{F} = (\mathbf{m}\alpha^{\mathbf{2}})\,(\dot{\mathbf{v}}_{\mathbf{2}} - \dot{\mathbf{v}}_{\mathbf{1}})$$

Analogues

Mechanical	Electrical
 $Y(s) = \frac{k}{s}$ $\frac{dF}{dt} = k(v_2 - v_1)$ spring	 $Y(s) = \frac{1}{Ls}$ $\frac{di}{dt} = \frac{1}{L}(v_2 - v_1)$ inductor
 $Y(s) = bs$ $F = b\frac{d(v_2 - v_1)}{dt}$ inerter	 $Y(s) = Cs$ $i = C\frac{d(v_2 - v_1)}{dt}$ capacitor
 $Y(s) = c$ $F = c(v_2 - v_1)$ damper	 $Y(s) = \frac{1}{R}$ $i = \frac{1}{R}(v_2 - v_1)$ resistor

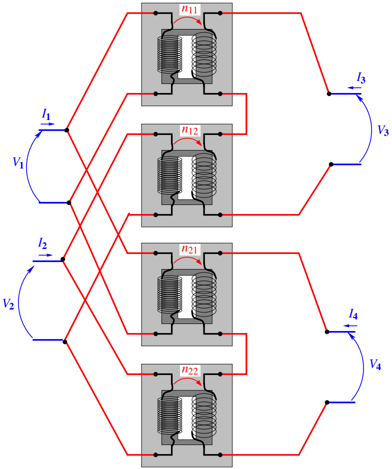
Network synthesis and state-space models

$$\left[\begin{array}{cc|cc} M_1 & M_2 & M_3 & M_4 \\ \hline \operatorname{diag} P_i\left(\frac{d}{dt}\right) & \operatorname{diag} Q_i\left(\frac{d}{dt}\right) & 0 & 0 \end{array}\right]\left[\begin{array}{c} i_{\text{int}} \\ v_{\text{int}} \\ \hline i_{\text{ext}} \\ v_{\text{ext}} \end{array}\right] = 0$$

Behavioral state-space model:

$$\mathscr{B}_s = \left\{(x, u, y) : \frac{d}{dt}x = Ax + Bu, y = Cx + Du\right\}$$

Transformer Synthesis



$$\begin{bmatrix} V_1 \\ V_2 \\ -- \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & n_{11} & n_{12} \\ 0 & 0 & n_{21} & n_{22} \\ -n_{11} & -n_{21} & 0 & 0 \\ -n_{12} & -n_{22} & 0 & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ -- \\ V_3 \\ V_4 \end{bmatrix}$$

Transformer Synthesis

$$\left[\begin{array}{cc|cc} 0 & 0 & N^T & -I \\ I & N & 0 & 0 \end{array}\right] \left[\begin{array}{c} i_a \\ i_b \\ \hline v_a \\ v_b \end{array}\right] = 0$$

Transformer and Gyrator Synthesis

Can synthesise

$$\left[\begin{array}{c|c} R & -I \end{array}\right] \left[\begin{array}{c} i \\ \hline v \end{array}\right] = 0$$

for any $R = -R^T$.

Transformer and Gyrator Synthesis

Can synthesise

$$\left[\begin{array}{c|c} R & -I \end{array}\right] \left[\begin{array}{c} i \\ \hline v \end{array}\right] = 0$$

for any $R = -R^T$.

Factor:

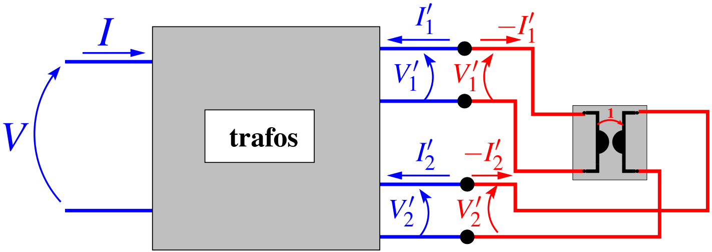
$$R = N^T \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} N$$

Transformer and Gyrator Synthesis

Synthesise with transformers: $$\left[\begin{array}{cc|cc} 0 & 0 & N^T & -I \\ I & N & 0 & 0 \end{array}\right] \left[\begin{array}{c} i' \\ i \\ \hline v' \\ v \end{array}\right] = 0$$

Synthesise with gyrators: $$\left[\begin{array}{cc|cc} 0 & -I & I & 0 \\ I & 0 & 0 & I \end{array}\right] \left[\begin{array}{c} i'' \\ \hline v'' \end{array}\right] = 0$$

Transformer and Gyrator Synthesis



Transformer and Gyrator Synthesis

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$$v = N^T v''$$

Transformer and Gyrator Synthesis

Synthesise with transformers: $$\left[\begin{array}{cc|cc} 0 & 0 & N^T & -I \\ I & N & 0 & 0 \end{array}\right] \left[\begin{array}{c} i' \\ i \\ \hline v' \\ v \end{array}\right] = 0$$

Synthesise with gyrators: $$\left[\begin{array}{cc|cc} 0 & -I & I & 0 \\ I & 0 & 0 & I \end{array}\right] \left[\begin{array}{c} i'' \\ \hline v'' \end{array}\right] = 0$$

$$v = -N^T \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} i''$$

Transformer and Gyrator Synthesis

Synthesise with transformers: $$\left[\begin{array}{cc|cc} 0 & 0 & N^T & -I \\ I & N & 0 & 0 \end{array}\right] \left[\begin{array}{c} i' \\ i \\ \hline v' \\ v \end{array}\right] = 0$$

Synthesise with gyrators: $$\left[\begin{array}{cc|cc} 0 & -I & I & 0 \\ I & 0 & 0 & I \end{array}\right] \left[\begin{array}{c} i'' \\ \hline v'' \end{array}\right] = 0$$

$$v = N^T \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} i'$$

Transformer and Gyrator Synthesis

Synthesise with transformers: $$\left[\begin{array}{cc|cc} 0 & 0 & N^T & -I \\ I & N & 0 & 0 \end{array}\right] \left[\begin{array}{c} i' \\ i \\ \hline v' \\ v \end{array}\right] = 0$$

Synthesise with gyrators: $$\left[\begin{array}{cc|cc} 0 & -I & I & 0 \\ I & 0 & 0 & I \end{array}\right] \left[\begin{array}{c} i'' \\ \hline v'' \end{array}\right] = 0$$

$$v = -N^T \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} Ni = Ri$$

Resistor, Transformer, and Gyrator Synthesis...

Can synthesise

$$\left[\begin{array}{c|c} R & -I \end{array}\right] \left[\begin{array}{c} i \\ \hline v \end{array}\right] = 0$$

for any R such that $R + R^T \succeq 0$.

Resistor, Transformer, and Gyrator Synthesis...

Can synthesise

$$\left[\begin{array}{c|c} R & -I \end{array}\right]\left[\begin{array}{c} i \\ \hline v \end{array}\right] = 0$$

for any R such that $R + R^T \succeq 0$.
Factor:

$$\frac{1}{2}\left(R + R^T\right) = S^T S$$

Resistor, Transformer, and Gyrator Synthesis...

Synth with trans and gyr: $\left[\begin{array}{cc|cc} 0 & S & I & 0 \\ -S^T & \frac{1}{2}\left(R^T - R\right) & 0 & I \end{array}\right] \left[\begin{array}{c} i' \\ i \\ \hline v' \\ v \end{array}\right] = 0$

Synthesise with resistors: $\left[\begin{array}{c|c} -I & I \end{array}\right] \left[\begin{array}{c} i'' \\ \hline v'' \end{array}\right] = 0$

RLTG synthesis

Synthesise with RTG:

$$\left[\begin{array}{cc|cc} -A & -I & -B & 0 \\ C & 0 & D & -I \end{array}\right] \left[\begin{array}{c} i_{\text{int}} \\ v_{\text{int}} \\ \hline i_{\text{ext}} \\ v_{\text{ext}} \end{array}\right] = 0$$

Synthesise with L

$$\left[\begin{array}{c|c} I\frac{d}{dt} & -I \end{array}\right] \left[\begin{array}{c} i' \\ \hline v' \end{array}\right] = 0$$

RLTG synthesis

$$\left[\begin{array}{cc|cc} -A & -I & -B & 0 \\ C & 0 & D & -I \\ \hline I\frac{d}{dt} & I & 0 & 0 \end{array}\right] \left[\begin{array}{c} i_{\text{int}} \\ v_{\text{int}} \\ \hline i_{\text{ext}} \\ v_{\text{ext}} \end{array}\right] = 0$$

RLTG synthesis

$$\left[\begin{array}{cc|cc} I\frac{d}{dt}-A & 0 & -B & 0 \\ C & 0 & D & -I \\ \hline I\frac{d}{dt} & I & 0 & 0 \end{array}\right]\left[\begin{array}{c} i_{\text{int}} \\ v_{\text{int}} \\ \hline i_{\text{ext}} \\ v_{\text{ext}} \end{array}\right]=0$$

RLTG synthesis

$$\left[\begin{array}{ccc|c} I\frac{d}{dt}-A & -B & 0 & 0 \\ C & D & -I & 0 \\ \hline I\frac{d}{dt} & 0 & 0 & I \end{array}\right]\left[\begin{array}{c} x \\ i_{\text{ext}} \\ v_{\text{ext}} \\ \hline v_{\text{int}} \end{array}\right] = 0$$

RLTG synthesis

Can synthesise any state-space model for which

$$\begin{bmatrix} -A & -B \\ C & D \end{bmatrix} + \begin{bmatrix} -A & -B \\ C & D \end{bmatrix}^T \succeq 0.$$

An equivalent characterisation

Every external RLCTG network behavior admits a state-space realisation with

$$\begin{bmatrix} -A & -B \\ C & D \end{bmatrix} + \begin{bmatrix} -A & -B \\ C & D \end{bmatrix}^T \succeq 0.$$

Special cases

Lossless Networks	**T**	N/A	$\left[\begin{array}{c:cc} & & \\ \hdashline & 0 & D_{12} \\ & -D_{12}^T & 0 \end{array}\right]$	–
	LT	$-I$	$\left[\begin{array}{c:cc} 0 & B_1 & 0 \\ \hdashline B_1^T & 0 & D_{12} \\ 0 & -D_{12}^T & 0 \end{array}\right]$	–
	CT	I	$\left[\begin{array}{c:cc} 0 & 0 & B_2 \\ \hdashline 0 & 0 & D_{12} \\ B_2^T & -D_{12}^T & 0 \end{array}\right]$	–
	LCT	$\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$	$\left[\begin{array}{cc:cc} 0 & A_{12} & 0 & B_{12} \\ -A_{12}^T & 0 & B_{21} & 0 \\ \hdashline 0 & B_{21}^T & 0 & D_{12} \\ B_{12}^T & 0 & -D_{12}^T & 0 \end{array}\right]$	–

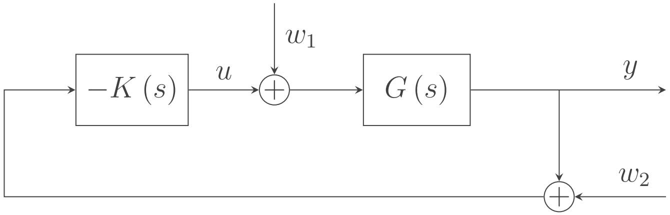
Special cases

Lossy Networks	**RT**	N/A	$\left[\begin{array}{c:cc} & & \\ \hdashline & D_{11} & D_{12} \\ & -D_{12}^T & D_{22} \end{array}\right]$	$D_{11} \succeq 0, D_{22} \succeq 0$
	RLT	$-I$	$\left[\begin{array}{c:cc} A & B_1 & B_2 \\ \hdashline B_1^T & D_{11} & D_{12} \\ -B_2^T & -D_{12}^T & D_{22} \end{array}\right]$	$\begin{bmatrix} -A & B_2 \\ B_2^T & D_{22} \end{bmatrix} \succeq 0, D_{11} \succeq 0$
	RCT	I	$\left[\begin{array}{c:cc} A & B_1 & B_2 \\ \hdashline -B_1^T & D_{11} & D_{12} \\ B_2^T & -D_{12}^T & D_{22} \end{array}\right]$	$\begin{bmatrix} -A & B_1 \\ B_1^T & D_{11} \end{bmatrix} \succeq 0, D_{22} \succeq 0$

Open problems

- Transformerless synthesis
- Synthesise resistive ’4 ports’
- ...

Exploit structure for Optimal Control

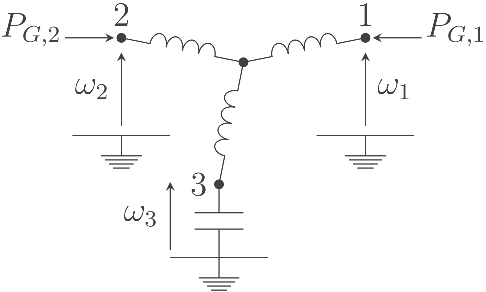


Exploit structure for Optimal Control

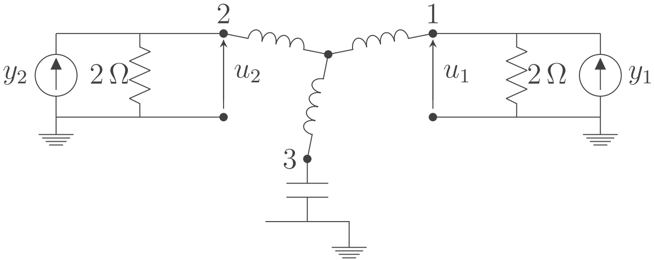
Optimal Controller:

$$\left[\begin{array}{c:c} A_c & B_c \\ \hdashline C_c & D_c \end{array}\right] = \begin{cases} \left[\begin{array}{c:c} A - 2BB^T & B \\ \hdashline B^T & 0 \end{array}\right] & \text{in the } H_2 \text{ case;} \\ \left[\begin{array}{:c} \hdashline \sqrt{2}I \end{array}\right] & \text{in the } H_\infty \text{ case;} \end{cases}$$

Example I



Example I



Example I

Reasonable starting point for grid forming inverter design:

1. Easy to scale
2. Inherits network structure
3. Nothing to do with sparsity!

Example II

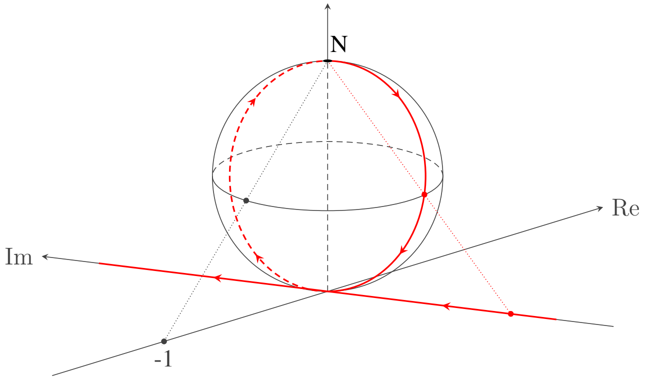


Figure 4: Illustration of the observation about the optimal control law from Remark 3. In the scalar case, the H_∞ norm of

$$\begin{bmatrix} G(s) \\ I \end{bmatrix} (I + K(s)G(s))^{-1} \begin{bmatrix} K(s) & I \end{bmatrix},$$

Example II

The objective of constrained least squares is to find an $\bar{x} \in \mathbb{R}^n$ that satisfies

$$\min_{\bar{x} \in \mathbb{R}^n} \|A\bar{x} - b\|_2 \, , \text{s.t.} \, C\bar{x} = d, \tag{33}$$

where $A \in \mathbb{R}^{m \times n}$, $C \in \mathbb{R}^{p \times n}$, $b \in \mathbb{R}^m$ and $d \in \mathbb{R}^p$ are the problem data. Constrained least squares encompasses a very broad class of problems including, for example, finite horizon LQR, and includes standard least squares and minimum norm solutions to a set of linear equations as special cases ($p = 0$, and $A = I$ and $b = 0$, respectively). The solution to eq. (33) can be obtained from the Karush-Kuhn-Tucker conditions

$$\begin{bmatrix} -A^T A & -C^T \\ C & 0 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{z} \end{bmatrix} = \begin{bmatrix} -A^T b \\ d \end{bmatrix}, \tag{34}$$

Example II

To this end, consider the system

$$\frac{d}{dt}x = \begin{bmatrix} 0 & -C^T \\ C & 0 \end{bmatrix} x + \begin{bmatrix} A^T \\ 0 \end{bmatrix} (u + w_1 - r_1) + \begin{bmatrix} 0 \\ I \end{bmatrix} r_2,$$
$$y = \begin{bmatrix} A & 0 \end{bmatrix} x + w_2. \tag{35}$$

Example II

eq. (35) that the closed loop system becomes

$$\frac{d}{dt}x = \begin{bmatrix} -A^T A & -C^T \\ C & 0 \end{bmatrix} x + \begin{bmatrix} A^T \\ 0 \end{bmatrix} (w_1 - r_1) + \begin{bmatrix} 0 \\ I \end{bmatrix} r_2,$$
$$y = \begin{bmatrix} A & 0 \end{bmatrix} x + w_2.$$

Therefore by applying the step inputs $r_1 = bH(t)$ and $r_2 = dH(t)$, where $H(t)$ denotes the unit step, we see that

$$\lim_{t\to\infty} x(t) = \begin{bmatrix} \bar{x} \\ \bar{z} \end{bmatrix}.$$