Lund University

Department of
**AUTOMATIC CONTROL**

# Exam FRTF01 - Physiological Models and Computation

**January 13 2020, 8-13**

### Points and grades

All answers must include a clear motivation. The total number of points is 25. The maximum number of points is specified for each subproblem. Preliminary grades:

Grade 3: 12 – 16.5 points
4: 17 – 21.5 points
5: 22 – 25 points

### Accepted aid

Lecture slides, any books (without relevant exercises with solutions), standard mathematical tables and "Formelsamling i reglerteknik". Calculator.

### Results

The result of the exam will be posted in LADOK no later than January 27. Information on when the corrected exam papers will be shown, will be given on the course homepage.

**1.** Consider the following state-space model

$$\dot{x} = \begin{bmatrix} -2 & -1 \\ 3 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

**a.** Calculate the transfer function and determine the static gain. (1 p)

**b.** Insert output feedback with a P controller $u(t) = k(r - y)$. Sketch the block diagram and calculate the closed-loop transfer function. (1.5 p)

**c.** For which values of $k$ is the closed-loop system asymptotically stable? (1.5 p)

**d.** Choose $k$ such that the closed-loop static gain becomes 0.25. (1 p)

*Solution*

**a.** The transfer function is calculated using the formula

$$G(s) = C\,(sI - A)^{-1}\,B + D$$

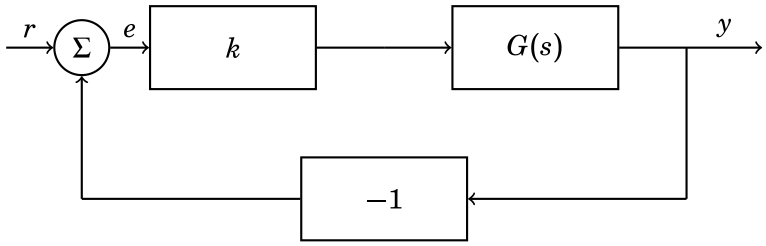
which in our case yields

$$G(s) = \frac{s-1}{s^2+s+1}.$$

The static gain can be calculated from the final value theorem by letting $u(t)$ be a unit step

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} sG(s)U(s)$$
$$\Rightarrow G(0) = -1$$

**b.** The block diagram of the closed loop system becomes




its transfer function can be calculated as

$$Y(s) = G(s)k(R(s) - Y(s))$$
$$\Rightarrow Y(s) = \frac{kG(s)}{1+kG(s)}R(s)$$
$$\Rightarrow G_{cl}(s) = \frac{kG(s)}{1+kG(s)} = \frac{k(s-1)}{s^2+(k+1)s+1-k}$$

**c.** The closed-loop transfer function is asymptotically stable if its poles has negative real part. For a second order characteristic polynomial $s^2 + as + b$ this occurs when $a, b > 0$. Thus $-1 < k < 1$ for the closed-loop system to be asymptotically stable.

**d.** The closed-loop static gain can be calculated using the final value theorem by letting $r(t)$ be a unit step.

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} sG_{cl}(s)R(s)$$
$$\Rightarrow G_{cl}(0) = \frac{-k}{1-k} = 0.25$$

which gives that $k = -1/3$.

**2.**

**a.** Draw a 3-compartment model (gut, blood and tissue) that describe a drugs path through the body, including and input $u$ to the gut compartment and elimination from the blood compartment. (1 p)

Table 1: Compartment data for problem 2.a

| Parameter | Description |
|---|---|
| $V_G$ | Distribution volume gut [l] |
| $V_B$ | Distribution volume blood [l] |
| $V_T$ | Distribution volume tissue [l] |
| $k_{GB}$ | Kinetics coefficient, gut to blood [min$^{-1}$] |
| $k_{BT}$ | Kinetics coefficient, blood to tissue [min$^{-1}$] |
| $k_{TB}$ | Kinetics coefficient, tissue to blood [min$^{-1}$] |
| $k_{e,B}$ | Elimination coefficient, blood [min$^{-1}$] |

**b.** Build a state space model from the compartments in 2.a where the input $u$ is the rate at which the drug is added to the gut and the output $y$ is the measured concentration of the drug in the blood. (1 p)

**c.** A drug bolus dissolves in the gut according to the Noyes-Whitney equation,

$$\dot{q} = \frac{DA}{d}(C_s - C_b),$$

where $q$ is released drug mass. Assume the dissolution to be constant for the time range considered in this problem. What is the steady state value of the measured concentration of the drug in the blood? (2 p)

*Solution*

**a.** See Figure 1 for the compartment model.

**b.** We get

$$\dot{G} = -k_{GB}G + u$$
$$\dot{B} = k_{GB}G - (k_{BT} + k_{e,B})B + k_{TB}T$$
$$\dot{T} = k_{BT}B - k_{TB}T$$
$$y = \frac{1}{V_B}B$$

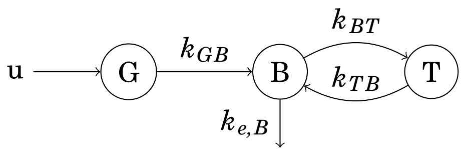



Figure 1: Compartment model for problem 2.a

or with $x = [G, B, T]^T$

$$\dot{x} = \begin{bmatrix} -k_{GB} & 0 & 0 \\ k_{GB} & -k_{BT} - k_{e,B} & k_{TB} \\ 0 & k_{BT} & -k_{TB} \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & \frac{1}{V_B} & 0 \end{bmatrix} x$$

**c.** We note that $u = \dot{q}$ since both represent the rate at which the drug is added to the gut. In steady state we have that $\dot{x} = 0$ and we can then see that $0 = \dot{G} + \dot{B} + \dot{T} = u - k_{e,B}B$.

$$u = \frac{DA}{d}(C_s - C_b)$$
$$B = \frac{1}{k_{e,B}}u = \frac{DA}{k_{e,B}d}(C_s - C_b)$$
$$y = \frac{1}{V_B}B = \frac{DA}{V_B k_{e,B} d}(C_s - C_b)$$

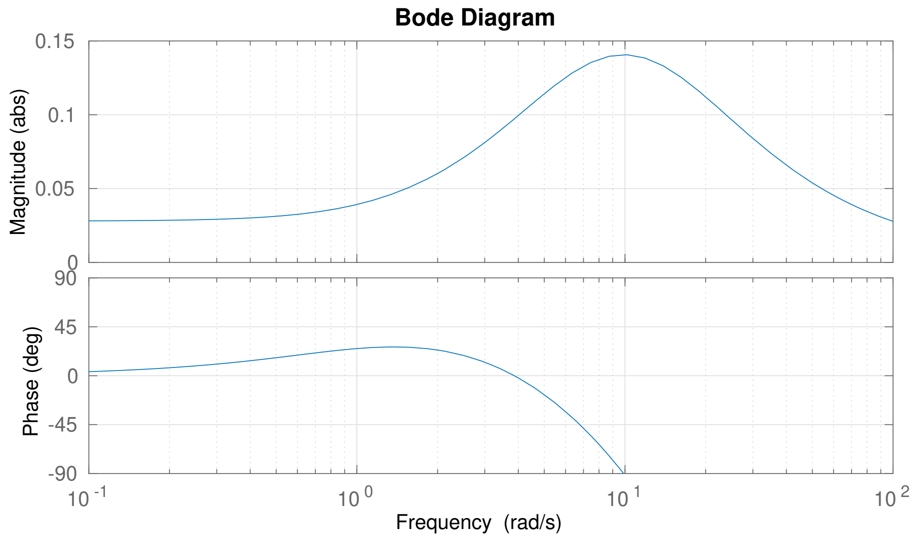



Figure 2: Bode diagram for problem 3.

**3.** The model of a certain biomedical apparatus has the following transfer function

$$G(s) = 2\frac{s+2}{(s+10)^2}e^{-0.2s}$$

describing the relation between input signal $u(t)$ and output signal $y(t)$.

**a.** Calculate $y(t)$ for the input $u(t) = \sin(4t)$. (2 p)

The model parameters are tweaked to provide a more accurate transfer function. The Bode diagram of the tweaked model can be seen in Figure 2.

**b.** Use this Bode diagram to determine the *new* output for the input $u(t) = \sin(4t)$. (1 p)

*Solution*

**a.** If the input is of the type $u(t) = \sin(wt)$, then the output becomes $y(t) = |G(iw)| \sin(wt + \arg G(iw))$. From our transfer function we get that

$$\begin{aligned}
|G(iw)| &= \frac{|i2w+4|}{|iw+10|^2}|e^{-0.2iw}| \\
&= \frac{\sqrt{4w^2+16}}{w^2+100} \\
\arg G(iw) &= \arg(i2w+4) - 2\arg(iw+10) - \arg\left(e^{-i0.2w}\right) \\
&= \tan^{-1}\left(\frac{2w}{4}\right) - 2\tan^{-1}\left(\frac{w}{10}\right) - 0.2w \\
\Rightarrow |G(iw = i4)| &= 0.0771 \\
\Rightarrow \arg G(iw = i4) &= -0.4539 \text{ rad/s}
\end{aligned}$$

which yields the output

$$y(t) = 0.0771\sin(4t - 0.4539)$$

**b.** The magnitude and phase can be read directly from the Bode diagram. For $w = 4$, we get that $y(t) = 0.1\sin(4t)$.

**4.** In this problem we consider the differential equation

$$\ddot{y} + \sqrt{y-1} - \dot{y}y = u^2$$

where $u$ is the input and $y$ is the output.

**a.** Introduce the variables $x_1 = y$ and $x_2 = \dot{y}$ and write the system on state space form. (1 p)

**b.** Find the stationary point for $u^0 = 1$. (1 p)

**c.** Linearize the system around the stationary point $(x_1^0, x_2^0, u^0)$ using the values you found in 4.b. (2 p)

*Solution*

**a.** A state-space model with $x_1 = y$ and $x_2 = \dot{y}$ is

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\sqrt{x_1 - 1} + x_1x_2 + u^2 \\
y &= x_1
\end{aligned}$$

**b.** Setting the derivatives to zero gives us the equations

$$\begin{aligned}
0 &= \dot{x}_1 = x_2^0 \\
0 &= \dot{x}_2 = -\sqrt{x_1^0 - 1} + x_1^0x_2^0 + (u^0)^2 = -\sqrt{x_1^0 - 1} + 1.
\end{aligned}$$

The solution of these equations gives us the stationary point $(x_1^0, x_2^0, u^0) = (2, 0, 1)$.

**c.** First calculate the partial derivatives.

$$\begin{aligned}
&\frac{\partial f_1}{\partial x_1} = 0 && \frac{\partial f_1}{\partial x_2} = 1 && \frac{\partial f_1}{\partial u} = 0 \\
&\frac{\partial f_2}{\partial x_1} = x_2^0 - \frac{1}{2\sqrt{x_1^0 - 1}} = -\frac{1}{2} && \frac{\partial f_2}{\partial x_2} = x_1^0 = 2 && \frac{\partial f_2}{\partial u} = 2u^0 = 2 \\
&\frac{\partial g}{\partial x_1} = 1 && \frac{\partial g}{\partial x_2} = 0 && \frac{\partial g}{\partial u} = 0
\end{aligned}$$

Then introduce the linearized variables $\Delta x = x - x^0$...

$$\begin{aligned}
\dot{\Delta x} &= \begin{bmatrix} 0 & 1 \\ -0.5 & 2 \end{bmatrix} \Delta x + \begin{bmatrix} 0 \\ 2 \end{bmatrix} \Delta u \\
\Delta u &= \begin{bmatrix} 1 & 0 \end{bmatrix} \Delta x
\end{aligned}$$

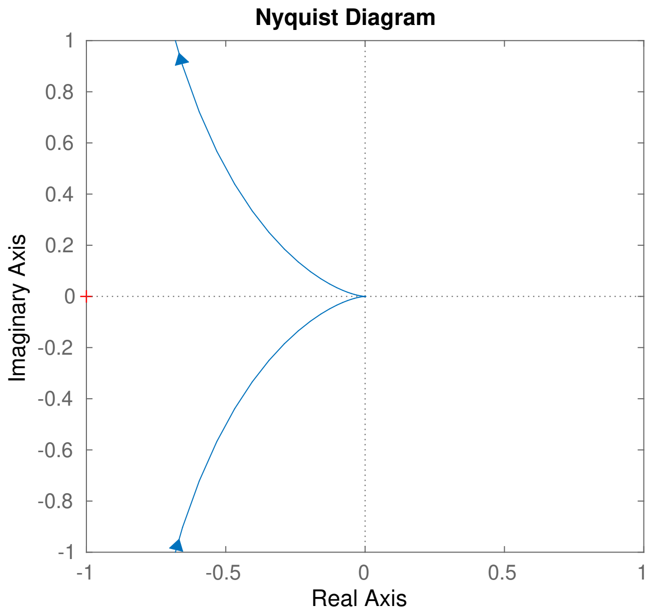



Figure 3: Nyquist diagram for problem 5.

**5.**

**a.** Find the gain and phase margins from the Nyquist diagram in Figure 3. Motivate your answer. (2 p)

**b.** Which of these transfer functions is the source of the Nyquist diagram in Figure 3? Motivate your answer. (2 p)

$$G_1(s) = \frac{1}{s+2}$$
$$G_2(s) = \frac{1}{s(s+1)}$$
$$G_3(s) = \frac{e^{-s}}{s+1}$$

*Solution*

**a.** The phase margin $\varphi_m = 51.8$ and is calculated as the angle between the negative real axis and the line through the origin and the point where the curve crosses the unit circle. The gain margin $A_m = \infty$ since the curve never crosses the negative real axis.

**b.** The correct answer is $G_2(s)$. Since one end of the curve disappears with seemingly infinite imaginary part we conclude it can't be a first order process $G_1(s)$, and neither a first order process with a time delay $G_3(s)$. There are more ways to motivate this.

**6.** A simplified version on how the glucose and insulin levels in the bloodstream depends on the food intake is given by the following model

$$\begin{aligned}\dot{G}_f(t) &= -k_{\text{food}} G_f(t) + u_f(t)\\ \dot{G}_b(t) &= k_{\text{food}} G_f(t) - k_{\text{ins}} I(t)\\ \dot{I}(t) &= k_{\text{prod}} G_b(t) - k_{\text{half}} I(t)\end{aligned}$$

where $G_f(t)$ is the glucose level in the ingested food, $G_b(t)$ and $I(t)$ the glucose and insulin levels in the bloodstream and $u_f(t)$ the food intake.

People suffering from diabetes type I and II has reduced production of insulin, represented in the model with the parameter $k_{\text{prod}}$. For the conditions $k_{\text{prod}}$ lies within the interval stated in the table below.

| Condition | $k_{\text{prod}}$ interval |
|---|---|
| Diabetes type I | [0, 0.05] |
| Diabetes type II | [0.05, 0.2] |
| Healthy | [0.2, 0.4] |

The remaining parameters can be assumed to be $k_{\text{food}} = 0.1$, $k_{\text{ins}} = 0.1$, $k_{\text{half}} = 0.5$.

**a.** Find the intervals where the poles reside for the system corresponding to a patient with diabetes type II. (1.5 p)

**Hint:** The system has one pole in $s = -0.1$, independent on the value $k_{\text{prod}}$.

Glucose levels in patients can be controlled via an artificial pancreas. Since we cannot affect $G_f(t)$ by changing the insulin level, the model can be simplified by disregarding this state, yielding

$$\begin{aligned}\dot{G}_b(t) &= -k_{\text{ins}} I(t) + u_b(t)\\ \dot{I}(t) &= k_{\text{prod}} G_b(t) - k_{\text{half}} I(t) + u_I(t)\end{aligned}$$

**b.** Introduce an artificial pancreas as a state feedback controller, such that a patient with $k_{\text{prod}} = 0.1$ gets the same poles as an healthy person with $k_{\text{prod}} = 0.3$. You may assume that $u_b(t) = 0$. (1.5 p)

**c.** Assume that we can only measure the glucose level in the blood, $y(t) = G_b(t)$, but would want to know the insulin level of our patient with $k_{\text{prod}} = 0.1$. Introduce the state estimator

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + K(y(t) - C\hat{x}(t))$$

and choose $K$ such that the poles for the dynamics of the estimation error $\tilde{x}(t) = x(t) - \hat{x}(t)$ becomes $p = (-0.1, -0.6)$. (2 p)

*Solution*

**a.** The state-space form of the model is

$$\dot{x} = \begin{bmatrix} -0.1 & 0 & 0 \\ 0.1 & 0 & -0.1 \\ 0 & k_{\text{prod}} & -0.5 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u_f$$

Calculating the determinant $sI - A$ yields the characteristic polynomial

$$\det(sI - A) = s^3 + 0.6s^2 + (0.05 + 0.1k_{\text{prod}})s + 0.01k_{\text{prod}}.$$

From the hint it is known that $s = -1$ is a pole. The polynomial can thus be factorized as

$$(s+1)(s^2 + as + b) = s^3 + 0.6s^2 + (0.05 + 0.1k_{\text{prod}})s + 0.01k_{\text{prod}}$$

which gives $a = 0.5$ and $b = 0.1k_{\text{prod}}$. The remaining poles thus becomes

$$p_{2,3} = -0.25 \pm \sqrt{\frac{0.25}{4} - 0.1k_{\text{prod}}}$$

which gives the pole intervals

| Condition | $k_{\text{prod}}$ interval | pole interval |
|---|---|---|
| Diabetes type II | [0.05, 0.2] | $\begin{cases} p_1 = -0.1 \\ p_2 = [-0.0438, -0.0102] \\ p_3 = [-0.4898, -0.4562] \end{cases}$ |

**b.** The new state-space representation becomes

$$\dot{x} = \begin{bmatrix} 0 & -0.1 \\ k_{\text{prod}} & -0.5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

Setting $k_{\text{prod}} = 0.1$ and inserting the state feedback controller $u(t) = -Lx + l_r r$ gives that

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & -0.1 \\ 0.1 & -0.5 \end{bmatrix} x - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} l_1 & l_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} l_r r \\ \Rightarrow \dot{x} &= \begin{bmatrix} 0 & -0.1 \\ 0.1 - l_1 & -0.5 - l_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} l_r r \end{aligned}$$

**easy way:** It is easily argued that letting $L = [-0.2, 0]$ gives the desired value of $k_{\text{prod}}$.

**hard way:** The healthy person has poles in $p = (-0.0697, -0.4303)$. The characteristic polynomial of the system under state feedback becomes

$$\det(sI - A + BL) = s^2 + (0.5 + l_2)s + 0.01 - 0.1l_1 = (s + 0.0697)(s + 0.4303)$$

which yields $L = [-0.2, 0]$.

**c.** The dynamics of the estimation error becomes

$$\begin{aligned}\dot{\tilde{x}} &= \dot{x} - \dot{\hat{x}} = Ax + Bu - A\hat{x} - Bu - K(Cx - C\hat{x}) \\ &= A(x - \hat{x}) - KC(x - \hat{x}) \\ &= (A - KC)\tilde{x}\end{aligned}$$

The characteristic polynomial becomes

$$\det(sI - A + KC) = s^2 + (0.5 + k_1)s + 0.5k_1 - 0.1k_2 + 0.01$$

which should have the same zeros as $(s + 0.1)(s + 0.6) = s^2 + 0.7s + 0.06$. Matching coefficients yields that $K = [0.2, 0.5]$.

**Good Luck!**