LUNDS UNIVERSITET

Institutionen för
REGLERTEKNIK

Reglerteknik AK

Exam 23 October 2023, 14-19

Points and Grades

All answers must include a clear motivation and a well-formulated answers. Answers may be given in English or Swedish. The total number of points is 25. The maximum number of points is specified for each subproblem.

Betyg :
Grade 3: at least 12 points
Grade 4: at least 17 points
Grade 5: at least 22 points

Accepted aids

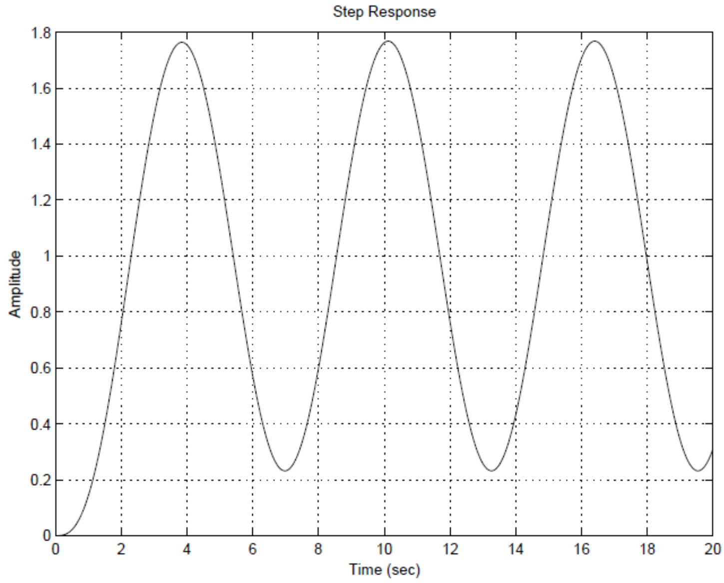
Standard mathematical tables (TEFYMA or equivalent), the authorized formula sheets in Reglerteknik AK. Pocket calculator without communication devices and that is not pre-programmed.

Results

The result of the exam will become accessible through LADOK. The solutions will be available in Canvas. An exam-viewing session will be announced in Canvas.

GOOD LUCK!

1. A PID controller should be designed for an unknown process. When the process is controlled by a proportional controller with $K = 0.6$ the step response in Fig. 1 is obtained. Use Ziegler-Nichols frequency method to design a PID controller. (2 p)



Figur 1: Step response for the closed loop system with $K = 0.6$ in Problem 1.

Solution

The period time is obtained from the graph and is $T_0 \approx 6$. In addition $K_0 = 0.6$. According to Ziegler-Nichols frequency method one obtains:

$$
\begin{aligned}
K &= 0.6K_0 = 0.36 \\
T_i &= T_0/2 = 3 \\
T_d &= T_0/8 = 0.75
\end{aligned}
$$

2. Six different Nyquist-diagrams are shown in Figure 2. Match these with the transfer functions below. Full motivation is required. (TIP: think in Bode plot terms) (3 p)

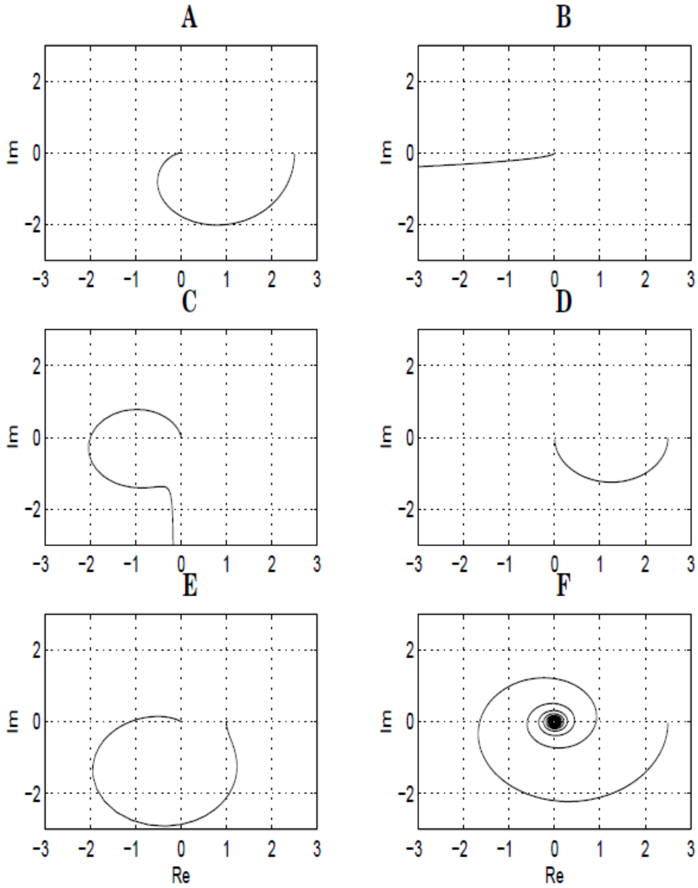
$$
G_1(s) = \frac{5}{s+2} \qquad G_4(s) = \frac{s+10}{2s^2}
$$

$$
G_2(s) = \frac{5}{s^2+2s+2} \qquad G_5(s) = \frac{5}{s+2}e^{-s}
$$

$$
G_3(s) = \frac{0.2}{s(s^2+0.2s+0.5)} \qquad G_6(s) = \frac{s+5}{10(s+1)(s^2+0.2s+0.5)}
$$

Solution

$G_1(s)$ is a first order system, i.e., the Nyquist curve lies entirely in the fourth quadrant, and has static gain 2.5. The only curve that matches this is D, i.e., $G_1(s) \rightarrow D$.



Figur 2: Nyquist-diagrams for Problem 2.

$G_2(s)$ is a second order system with static gain 2.5. The curve should start in 2.5, continue first in the fourth quadrant and then in the third. It should approach the origin along the negative real axis when $\omega \to \infty$. The only curve that matches this is A, i.e., $G_2(s) \to A$.

$G_3(s)$ is a third order system with one integrator pole. For small ω The curve lies along the negative imaginary axis (phase $= -90°$). The only curve that matches this is C, i.e., $G_3(s) \to C$.

$G_4(s)$ is a second order system with two integrators. For small ω the curve comes along the negative real axis (phase $= -180°$). The only curve that matches this is B, i.e., $G_4(s) \to B$.

$G_5(s)$ contains a time delay, i.e., the curve will spiral towards the origin Hence, $G_5(s) \to F$.

$G_6(s)$ has one real pole, two complex conjugated poles and one real zero. The phase should go from $0°$ to $-180°$, but can temporarily be less than $-180°$. This corresponds curve E, i.e., $G_6(s) \to E$.

3.

a. Assume that you are using a PID controller in standard form, i.e.,

$$U(s) = K(E(s) + \frac{1}{T_I s}E(s) + T_D s E(s))$$

to control the level in the lower tank in Lab 2. Assume further that the control loop is in stationarity without any disturbances and that the control signal has not hit any limitations. Which (one or several) of the terms in the controller, i.e., the P-term, the I-term, and the D-term, contribute to the control signal, $u(t)$, of the controller? (1 p)

b. Assume instead that you use a PID controller with setpoint weighting in the P-term, i.e., the P-term now is $K(bR(s) - Y(s))$, and that $b \neq 1$. Under the same assumptions, which (one or several) of the terms now contribute to the control signal? (1 p)

Solution

a. In stationarity all signals are constant, including the reference signal. Since we have integral action in the controller the stationary error and the derivative of the error are both zero, i.e., both the P-term and the D-term will be zero and it is only the I-term that contributes to the control signal.

b. When $b \neq 1$ the P-term is different from zero also when the error is zero. Hence, the control signal is obtained by the P-term and the I-term.

4. Given the following state space description

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

a. Determine a control law

$$u = k_r r - Kx$$

for the system such that the poles of the closed loop system are placed in -11 and the stationary gain is 1 between r and y. (2 p)

b. Determine a Kalman filter

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + L(y - C\hat{x})$$

for the system such that the poles of the observer are at -19. (1 p)

Solution

a. If $u = k_r r - Kx$ the system becomes.

$$\dot{x} = Ax + B(k_r r - Kx) = (A - BK)x + Bk_r r$$

K should be chosen so that the eigenvalues of $(A - BK)$ are in -11. The eigenvalues of $(A - BK)$ are calculated by solving the characteristic equation

$$det(sI - A + BK) = det \begin{pmatrix} s + 2 + 2k_1 & 2k_2 \\ -2 & s + 3 \end{pmatrix} = (s + 2 + 2k_1)(s + 3) + 4k_2 =$$

$$= s^2 + (5+2k_1)s + 6 + 6k_1 + 4k_2 = 0$$

If the requested eigenvalues should be places at -11 the characteristic equation should equal to

$$(s+11)^2 = s^2 + 22s + 121$$

$$= s^2 + (5+2k_1)s + 6 + 6k_1 + 4k_2 = s^2 + 22s + 121$$

This gives two equations namely

$$5 + 2k_1 = 22$$

$$6 + 6k_1 + 4k_2 = 121$$

solving this system of equations gives $k_1 = 8.5$ and $k_2 = 16$.

k_r is calculated by constructing the transfer function between r and y and choosing k_r so that the static gains of this transfer function is 1. This is done by Laplace transforming the system.

$$sX = (A - BK)X + Bk_rR \rightarrow X = (sI - A + BK)^{-1}Bk_rR$$

$$Y = CX \rightarrow C(sI - A + BK)^{-1}Bk_rR$$

The transfer function from R to Y is

$$G(s) = C(sI - A + BK)^{-1}Bk_r$$

The static gain should 1 which means that

$$G(0) = C(-A + BK)^{-1}Bk_r = 1$$

$$k_r = \frac{1}{C(-A+BK)^{-1}B}$$

To find k_r one should calculate $C(-A + BK)^{-1}B$ and take the inverse of it.

$$C(-A+BK)^{-1}B = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 19 & 32 \\ -2 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \frac{4}{121}$$

$$k_r = \frac{121}{4} = 30.25$$

The control law is

$$u = 30.25r - \begin{pmatrix} 8.5 & 16 \end{pmatrix} x$$

b. The eigenvalues of the Kalman filter are calculated by calculating the eigenvalues of $(A - LC)$. This is done by solving the characteristic equation.

$$det(sI - A + LC) = det \begin{pmatrix} s+2 & l_1 \\ -2 & s+3+l_2 \end{pmatrix} = (s+2)(s+3+l_2) + 2l_1 =$$

$$s^2 + (5+l_2)s + 6 + 2l_2 + 2l_1 = 0$$

If the requested eigenvalues should be places at -19 the characteristic equation should equal to

$$(s+19)^2 = 2^2 + 38s + 361$$

$$s^2 + (5 + l_2)s + 6 + 2l_2 + 2l_1 = 2^2 + 38s + 361$$

This gives two equations namely

$$5 + l_2 = 38$$

$$6 + 2l_2 + 2l_1 = 361$$

solving this system of equations gives $l_1 = 144.5$ and $l_2 = 33$.

$$L = \begin{pmatrix} 114.5 \\ 33 \end{pmatrix}$$

5. Given the system

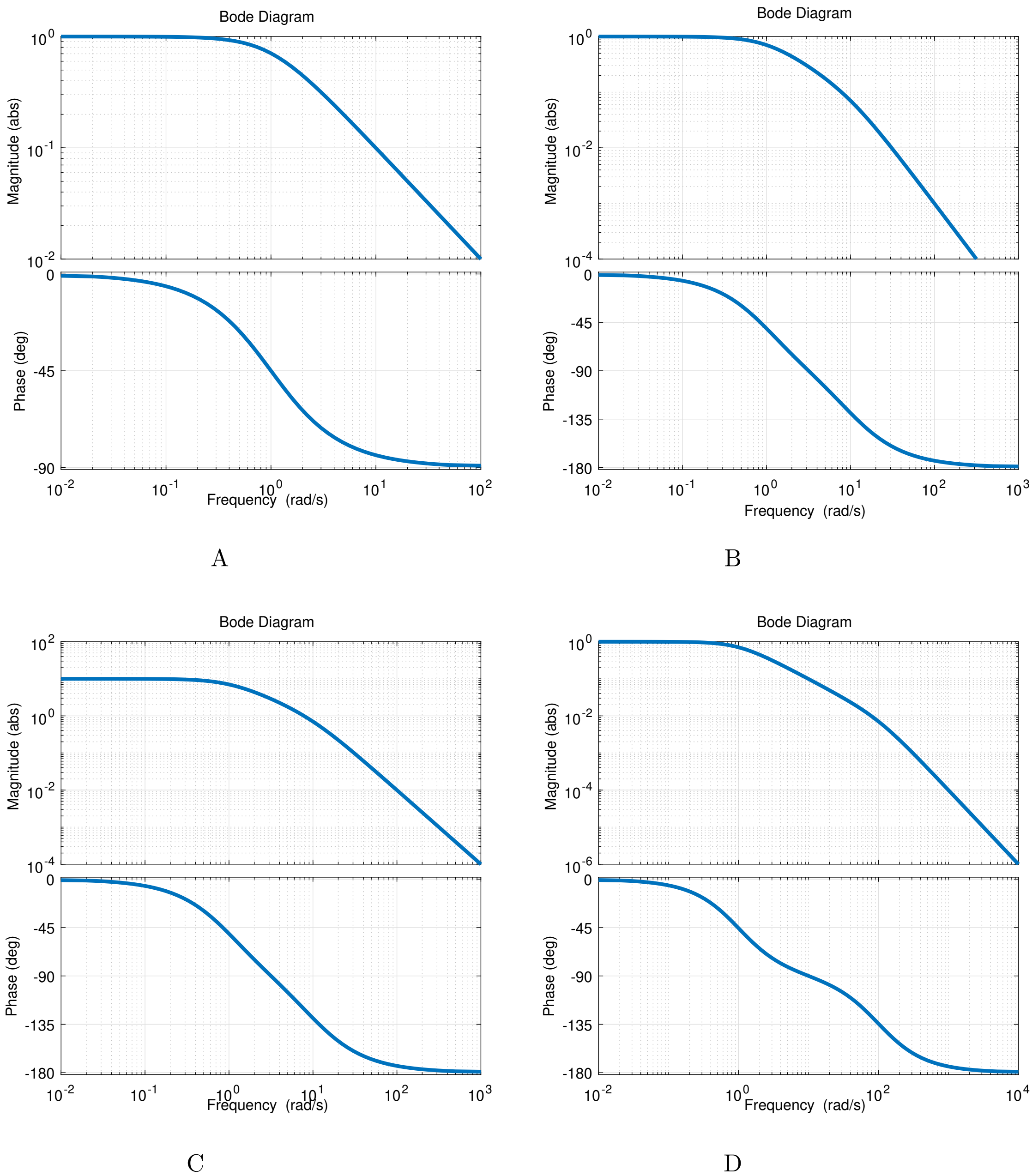
$$\frac{\ddot{y}}{10} = -y - \frac{11\dot{y}}{10} + u$$

a. Write the system as a transfer function (0.5 p)

b. Factorize the transfer function to obtain the following form. (1 p)

$$G(s) = \frac{c}{(as + 1)(bs + 1)}$$

c. Pick the right Bode plot for the system transfer function from the figures provided bellow [A-D] in figure 3. (1 p)



Figur 3: Bode plots for Problem 4.c.

1. **INFO**: For full credit you must motivate your answer.

d. What is the output signal if the input signal is $u = 3\sin(0.05t) + 2\sin(100t)$. (1.5 p)

1. **INFO**: Both calculations or using the Bode plot will give full credit. Pick the method you prefer.
2. **INFO**: If you have not succeeded in finding/factoring the transfer function or finding the right Bode plot, use the following transfer function instead. Please note that this is not the answer one obtains from the problem

formulation i.e you can not use the Bode plot:

$$G(s) = \frac{1}{(\frac{s}{11}+1)(s+1)}$$

3. **TIP**:Use the linearity property of a transfer function.

Solution

a. 1. Laplace transform $\frac{\ddot{y}}{10} = -y - \frac{11\dot{y}}{10} + u$

$$\frac{s^2Y}{10} = -Y - \frac{11sY}{10} + U$$

2. Move Y over to the left-hand side and isolate Y.

$$\left(\frac{s^2}{10} + 1 + \frac{11s}{10}\right) Y = U$$

3. Move everything except Y over to the right-hand side.

$$Y = \frac{1}{\left(\frac{s^2}{10} + 1 + \frac{11s}{10}\right)} U$$

$$G(s) = \frac{1}{\left(\frac{s^2}{10} + \frac{11s}{10} + 1\right)} = \frac{10}{s^2 + 11s + 10}$$

b. If we factorize the denominator polynomial $s^2 + 11s + 10$, i.e. solve the corresponding quadratic equation, we get

$$s_{1,2} = -\frac{11}{2} \pm \sqrt{(\frac{11}{2})^2 - 10},$$

i.e. $s = -10$ and $s = -1$. This means that

$$G(s) = \frac{10}{(s+10)(s+1)} = \frac{1}{(\frac{s}{10}+1)(s+1)},$$

i.e $c = 1$, $a = 0.1$, and $b = 1$.

c. The phase curve must go from 0° to −180°, i.e. diagram A is excluded. The gain at low frequencies is 1, i.e. diagram C is excluded. The amplitude curve must have two breaking frequencies at $\omega = 1$ rad/s and one at $\omega = 10$ rad/s. Chart D appears to have two break frequencies but these are at $\omega = 1$ rad/s and $\omega = 100$ rad/s. The only thing left is the Bode plot B.

d. 1. Given that the transfer function is linear, you can look at each input signal separately and then add their output.

2. If $u = 3\sin(0.05t)$ then we know that G(s) can be approximated as 1. Given that

$$\frac{1}{(\frac{i0.05}{10}+1)} \approx 1$$
$$\frac{1}{(i0.05+1)} \approx 1$$

That is, the gain is 1 and the phase shift becomes 0 so the output signal is $3\sin(0.05t)$. This can also be seen from the Bode plot.

3. If $u = 2\sin(100t)$

$$|G(iw)| = |\frac{1}{(\frac{iw}{10}+1)(iw+1)}|$$
$$|G(i100)| = |\frac{1}{(i10+1)(i100+1)}| = \frac{1}{\sqrt{10^2+1}\sqrt{100^2+1}} =$$
$$\frac{1}{\sqrt{101}\sqrt{10001}} \approx 0.001$$
$$arg(G(i100)) = arg(\frac{1}{(i10+1)(i100+1)}) = arg(\frac{1}{110i-999})$$
$$= arg(\frac{-999-110i}{110^2+999^2}) = arctan(\frac{110}{999}) - \pi = -3.03rad = -174°$$

4. The output signal will be $0.002\sin(100t - 3.03)$. This can also be seen from the Bode plot.
5. The final output will be

$$y = 3\sin(0.05t) + 0.002\sin(100t - 3.03)$$

ALTERNATIVE ANSWER:

1. Given that the transfer function is linear, you can look at each input signal separately and then add their output.
2. If $u = 3\sin(0.05t)$ then we know that G(s) can be approximated as 1. Given that

$$\frac{1}{(\frac{i0.05}{11}+1)} \approx 1$$
$$\frac{1}{(i0.05+1)} \approx 1$$

That is, the gain is 1 and the phase shift becomes 0 so the output signal is $3\sin(0.05t)$.

3. If $u = 2\sin(100t)$

$$|G(iw)| = |\frac{1}{(\frac{iw}{11}+1)(iw+1)}|$$
$$|G(i100)| = |\frac{1}{(\frac{i100}{11}+1)(i100+1)}| = \frac{1}{\sqrt{\frac{100^2}{11^2}+1}\sqrt{100^2+1}} =$$
$$\frac{1}{\sqrt{83.6}\sqrt{10001}} \approx 0.001$$
$$arg(G(i100)) = arg(\frac{1}{(\frac{i100}{11}+1)(i100+1)}) = arg(\frac{1}{109i-908})$$
$$= arg(\frac{-908-109i}{109^2+908^2}) = arctan(\frac{109}{908}) - \pi = -3.02rad = -173°$$

4. The output signal will be $0.002\sin(100t - 3.02)$.
5. The final output will be

$$y = 3\sin(0.05t) + 0.002\sin(100t - 3.02)$$

6. Consider the following linear state-space system.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} u$$
$$y = \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

a. Calculate the transfer function $G(s)$ for the system. (1 p)

b. Explain why you get the result that you get. (1 p)

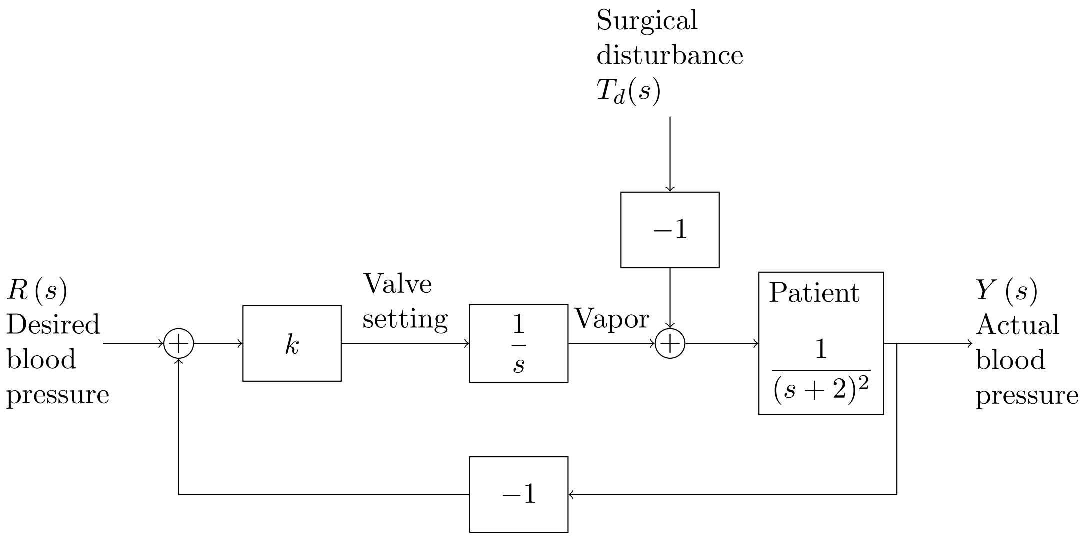
Solution

a.

$$\begin{aligned} G(s) &= C(sI - A)^{-1}B = \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} s+1 & 0 \\ 0 & s-2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 3 \end{bmatrix} \\ &= \frac{\begin{bmatrix} -1 & 0 \end{bmatrix}}{(s+1)(s-2)} \begin{bmatrix} s-2 & 0 \\ 0 & s+1 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \end{bmatrix} \\ &= \frac{0}{(s+1)(s-2)} = 0 \end{aligned}$$

b. Since the A-matrix is on diagonal form one can directly see that state x_1 is not controllable and state x_2 is not observable. Hence, none of the states are visible in the transfer function.

7. A system that controls the mean arterial pressure during anesthesia has been designed and tested. The level of arterial pressure is postulated to be a proxy for depth of anesthesia during surgery. A block diagram of the system is shown in Figure 4, where the impact of surgery is represented by the disturbance $T_d(s)$.



Figur 4: Block diagram for Problem 7.

a. Determine the transfer function from $T_d(s)$ to $Y(s)$. (1 p)

b. For what values of k is the system asymptotically stable? (1 p)

c. Determine the steady-state error due to a unit step disturbance. (1 p)

d. Determine the transfer function from the input $R(s)$ to $Y(s)$. (1 p)

e. Determine the steady-state error for a ramp input. (1 p)

f. What is the range of k for which we get a steady-state error less than 0.25 due to ramp inputs. (1 p)

Solution

a. The transfer function from the disturbance $T_d(s)$ to the output $Y(s)$ is found by setting $R(s) = 0$. We get

$$\frac{Y(s)}{T_d(s)} = \frac{-s}{s^3 + 4s^2 + 4s + k}.$$

b. The characteristic polynomial is of third order and takes the form $D(s) = s^3 + a_2 s^2 + a_1 s + a_0$. For asymptotic stability we require a_0, a_1, a_2 to be positive and also $a_2 a_1 > a_0$. This translates to $k > 0$ and $k < 16$ and hence k should be in the range $0 < k < 16$.

c. Assume k lies in the range $0 < k < 16$ we could use the final value theorem to obtain the steady state error. Having a unit step disturbance means $T_d(s) = \frac{1}{s}$. We get

$$e_{ss} = \lim_{s \to 0} s \left(0 + \frac{s}{s^3 + 4s^2 + 4s + k} \right) \frac{1}{s} = 0.$$

Note that in this case the desired output to a unit step disturbance is 0.

d. The closed-loop transfer function is

$$G(s) = \frac{Y(s)}{R(s)} = \frac{k}{s^3 + 4s^2 + 4s + k}.$$

e. The steady-state error when $R(s) = \frac{1}{s^2}$ using the final value theorem is given by

$$e_{ss} = \lim_{s \to 0} s\left(1 - G\left(s\right)\right)\frac{1}{s^2} = \frac{4}{k}.$$

This result is obtained by applying L'Hôpital's rule (by differentiating the numerator and denominator one ti me and then taking the limit).

f. If we want the error to be less than 0.25, this translates to $e_{ss} = \frac{4}{k} < 0.25 \leftrightarrow k > 16$. Intersecting this condition with the stability condition yields that there are no values of k such that this error reduction level is attained.

8. A process, denoted as P, is controlled by a controller, denoted as C. While the phase margin of the entire system is acceptable, the system's speed is too slow. Your task is to design a lead compensation link that will double the crossover frequency, w_c, while keeping the phase margin, φ_m, unchanged.

The system can be described as follows:

$$P = \frac{2}{s+1}\exp^{-0.25s} \quad , \quad C = \frac{1}{\sqrt{2}}$$

(3 p)

Solution

The original crossover frequency, denoted as w_c, is calculated using

$$|P(iw_c)C(iw_c)| = \left|\frac{2}{\sqrt{2}(iw_c+1)}\exp^{-0.25iw_c}\right| = 1$$

Since $\left|\frac{2}{\sqrt{2}}\right| = \frac{2}{\sqrt{2}}$ and $\left|\exp^{-0.25iw_c}\right| = 1$, we are left with:

$$\frac{2}{\sqrt{2}|(iw_c+1)|} = \frac{2}{\sqrt{2}\sqrt{w_c^2+1}} = 1$$

$$\frac{2}{\sqrt{2}} = \sqrt{w_c^2+1}$$

$$2 = w_c^2 + 1$$

$$w_c = 1$$

The original phase margin, denoted as φ_m, is calculated using the argument of $P(iw_c)C(iw_c)$.

$$\arg(P(iw_c)C(iw_c)) = \arg\left(\frac{2}{\sqrt{2}(iw_c+1)}\exp^{-0.25iw_c}\right) =$$

$$\arg\left(\frac{2}{\sqrt{2}(iw_c+1)}\right) + \arg\left(\exp^{-0.25iw_c}\right) = \arg\left(\frac{2-2iw_c}{\sqrt{2}(w_c^2+1)}\right) - 0.25w_c$$

$$\arctan(-w_c) - 0.25w_c$$

Insertion of the crossover frequency gives:

$$\arctan(-1) - 0.25 = -1.035 \text{ rad} \approx -60°$$

$$\varphi_m = 180 - 60 = 120°$$

To solve the problem.

1. Decide on the new crossover frequency, denoted as w_c^n, and on the new phase margin, denoted as φ_m^n.

$$w_c^n = 2w_c = 2$$

$$\varphi_m^n = \varphi_m$$

2. Calculate how much phase is in the new crossover frequency, calculate how much the phase there should be increased to meet the specifications given in step 1 and choose an N that gives this phase increase.
   (a) The phase at the new crossover frequency, denoted φ, is calculated using the exact same method as shown above:

$$\arctan(-w_c^n) - 0.25w_c^n = \arctan(-2) - 0.5 = -1.61 \approx -92°$$

$$\varphi = 180 - 92 = 88°$$

   (b) How much the phase should be increased, denoted $\Delta\varphi$, is calculated using:

$$\Delta\varphi = \varphi_m^n - \varphi = 120 - 88 = 32°$$

   (c) N is chosen to increase the phase. In this case, the phase should be increased by approximately 30°, which, with the help of the formula sheet, gives us N equal to 3.

3. Choose b so that the top of the newly created phase increase coincides with the new crossover frequency:

$$w_c^n = b\sqrt{N} \to b = \frac{w_c^n}{\sqrt{N}} = \frac{2}{\sqrt{3}} \approx 1.15$$

4. Choose K_k so that $|P(iw_c^n)C(iw_c^n)G_k(iw_c^n)| = 1$, where G_k is the lead compensation link and $|G_k(iw_c^n)| = K_k\sqrt{N}$.

$$|P(iw_c^n)C(iw_c^n)G_k(iw_c^n)| = |P(iw_c^n)C(iw_c^n)|K_k\sqrt{N} = 1$$

$$\frac{2}{\sqrt{2}\sqrt{(w_c^n)^2+1}}K_k\sqrt{N} = 1$$

$$K_k = \sqrt{\frac{(w_c^n)^2+1}{2N}} \approx 0.91$$

   So the final compensation link looks like:

$$G_k(s) = K_kN\frac{s+b}{s+bN} = 2.73\frac{s+1.15}{s+3.46}$$