

# **AUTOMATIC CONTROL**

## **Collection of Formulae**

Department of Automatic Control

Lund University

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# Matrix theory

## Notation

Matrix of order  $m \times n$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Vector of dimension  $n$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

## Transpose

$$B = A^T$$

$$b_{ij} = a_{ji}$$

$$(AB)^T = B^T A^T$$

The matrix is symmetric if  $a_{ij} = a_{ji}$ .

## Determinant

$$\det A = |A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

If  $A$  is of order  $2 \times 2$ , then

$$\det A = a_{11}a_{22} - a_{12}a_{21}$$

In general

$$\begin{aligned} \det A &= \sum_{i=1}^n a_{ij}(-1)^{i+j} \det M_{ij} \\ &= \sum_{j=1}^n a_{ij}(-1)^{i+j} \det M_{ij} \end{aligned}$$

where  $M_{ij}$  is the matrix one obtains if row  $i$  and column  $j$  are removed from the matrix  $A$ .

## Inverse

$$A^{-1}A = AA^{-1} = I \quad (\det A \neq 0)$$

If  $A$  is of order  $2 \times 2$ , then

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

In general,

$$A^{-1} = \frac{1}{\det A} C^T$$

where the elements in  $C$  are given by

$$c_{ij} = (-1)^{i+j} \det M_{ij}$$

### **Eigenvalues and eigenvectors**

The eigenvalues ( $\lambda_i, \quad i = 1, 2, \dots, n$ ) and the eigenvectors ( $x_i, \quad i = 1, 2, \dots, n$ ) are given as the solutions to the equation system

$$Ax = \lambda x$$

which has a solution if

$$\det(\lambda I - A) = \lambda^n + \alpha_1 \lambda^{n-1} + \alpha_2 \lambda^{n-2} + \dots + \alpha_n = 0$$

$\lambda^n + \alpha_1 \lambda^{n-1} + \alpha_2 \lambda^{n-2} + \dots + \alpha_n$  is called *the characteristic polynomial*.  
 $\det(\lambda I - A) = 0$  is called *the characteristic equation*.

## Dynamical systems

### State-space equations

$$\begin{aligned}\frac{dx}{dt} &= Ax + Bu \\ y &= Cx + Du \\ x(t) &= e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau\end{aligned}$$

### Weighting function

$$\begin{aligned}y(t) &= \int_0^t h(t-\tau)u(\tau)d\tau \\ h(t) &= Ce^{At}B + D\delta(t)\end{aligned}$$

### Transfer function

$$\begin{aligned}Y(s) &= G(s)U(s) \\ G(s) &= C(sI - A)^{-1}B + D = \mathcal{L}\{h(t)\}\end{aligned}$$

The denominator of  $G$  is the characteristic polynomial to the matrix  $A$ .

### Frequency response

$$\begin{aligned}u(t) &= \sin \omega t \\ y(t) &= a \sin(\omega t + \varphi)\end{aligned}$$

$$\begin{aligned}a &= |G(i\omega)| \\ \varphi &= \arg G(i\omega)\end{aligned}$$

## Linearization

If the nonlinear system

$$\begin{aligned}\frac{dx}{dt} &= f(x, u) \\ y &= g(x, u)\end{aligned}$$

is linearized around a stationary point  $(x_0, u_0)$ , a change of variables

$$\begin{aligned}\Delta x &= x - x_0 \\ \Delta u &= u - u_0 \\ \Delta y &= y - y_0\end{aligned}$$

then gives the linear system

$$\begin{aligned}\frac{d\Delta x}{dt} &= \frac{\partial f}{\partial x}(x_0, u_0)\Delta x + \frac{\partial f}{\partial u}(x_0, u_0)\Delta u \\ \Delta y &= \frac{\partial g}{\partial x}(x_0, u_0)\Delta x + \frac{\partial g}{\partial u}(x_0, u_0)\Delta u\end{aligned}$$

## State-space representations

### 1. Diagonal form

$$\frac{dz}{dt} = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} z + \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix} u$$

$$y = \begin{pmatrix} \gamma_1 & \gamma_2 & \dots & \gamma_n \end{pmatrix} z + Du$$

### 2. Observable canonical form

$$\frac{dz}{dt} = \begin{pmatrix} -a_1 & 1 & 0 & \dots & 0 \\ -a_2 & 0 & 1 & & 0 \\ \vdots & & & & \\ -a_{n-1} & 0 & 0 & & 1 \\ -a_n & 0 & 0 & & 0 \end{pmatrix} z + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \end{pmatrix} z + Du$$

### 3. Controllable canonical form

$$\frac{dz}{dt} = \begin{pmatrix} -a_1 & -a_2 & \dots & -a_{n-1} & -a_n \\ 1 & 0 & & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & & 1 & 0 \end{pmatrix} z + \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} b_1 & b_2 & \dots & b_n \end{pmatrix} z + Du$$

The transfer function of the system is

$$G(s) = D + \frac{b_1 s^{n-1} + b_2 s^{n-2} + \dots + b_n}{s^n + a_1 s^{n-1} + \dots + a_n}$$

$$= D + \frac{\beta_1 \gamma_1}{s - \lambda_1} + \frac{\beta_2 \gamma_2}{s - \lambda_2} + \dots + \frac{\beta_n \gamma_n}{s - \lambda_n}$$

# The Laplace transform

## Operator lexicon

	Laplace transform $F(s)$	Time function $f(t)$	
1	$\alpha F_1(s) + \beta F_2(s)$	$\alpha f_1(t) + \beta f_2(t)$	Linearity
2	$F(s + a)$	$e^{-at} f(t)$	Damping
3	$e^{-as} F(s)$	$\begin{cases} f(t - a) & t - a > 0 \\ 0 & t - a < 0 \end{cases}$	Time delay
4	$\frac{1}{a} F\left(\frac{s}{a}\right) \quad (a > 0)$	$f(at)$	Scaling in $t$ -domain
5	$F(as) \quad (a > 0)$	$\frac{1}{a} f\left(\frac{t}{a}\right)$	Scaling in $s$ -domain
6	$F_1(s)F_2(s)$	$\int_0^t f_1(t - \tau)f_2(\tau) d\tau$	Convolution in $t$ -domain
7	$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F_1(\sigma)F_2(s - \sigma) d\sigma$	$f_1(t)f_2(t)$	Convolution in $s$ -domain
8	$sF(s) - f(0)$	$f'(t)$	Differentiation in $t$ -domain
9	$s^2 F(s) - sf(0) - f'(0)$	$f''(t)$	
10	$s^n F(s) - s^{n-1}f(0) - \dots - f^{(n-1)}(0)$	$f^{(n)}(t)$	
11	$\frac{d^n F(s)}{ds^n}$	$(-t)^n f(t)$	Differentiation in $s$ -domain
12	$\frac{1}{s} F(s)$	$\int_0^t f(\tau) d\tau$	Integration in $t$ -domain
13	$\int_s^\infty F(\sigma) d\sigma$	$\frac{f(t)}{t}$	Integration in $s$ -domain
14	$\lim_{s \rightarrow 0} sF(s)$	$\lim_{t \rightarrow \infty} f(t)$	Final value theorem
15	$\lim_{s \rightarrow \infty} sF(s)$	$\lim_{t \rightarrow 0} f(t)$	Initial value theorem

### Transform lexicon

	Laplace transform $F(s)$	Time function $f(t)$	
1	1	$\delta(t)$	Dirac function
2	$\frac{1}{s}$	1	Step function
3	$\frac{1}{s^2}$	$t$	Ramp function
4	$\frac{1}{s^3}$	$\frac{1}{2} t^2$	Acceleration
5	$\frac{1}{s^{n+1}}$	$\frac{t^n}{n!}$	
6	$\frac{1}{s+a}$	$e^{-at}$	
7	$\frac{1}{(s+a)^2}$	$t \cdot e^{-at}$	
8	$\frac{s}{(s+a)^2}$	$(1-at)e^{-at}$	
9	$\frac{1}{1+sT}$	$\frac{1}{T} e^{-t/T}$	
10	$\frac{a}{s^2+a^2}$	$\sin at$	
11	$\frac{a}{s^2-a^2}$	$\sinh at$	
12	$\frac{s}{s^2+a^2}$	$\cos at$	
13	$\frac{s}{s^2-a^2}$	$\cosh at$	
14	$\frac{1}{s(s+a)}$	$\frac{1}{a} (1 - e^{-at})$	
15	$\frac{1}{s(1+sT)}$	$1 - e^{-t/T}$	
16	$\frac{1}{(s+a)(s+b)}$	$\frac{e^{-bt} - e^{-at}}{a-b}$	

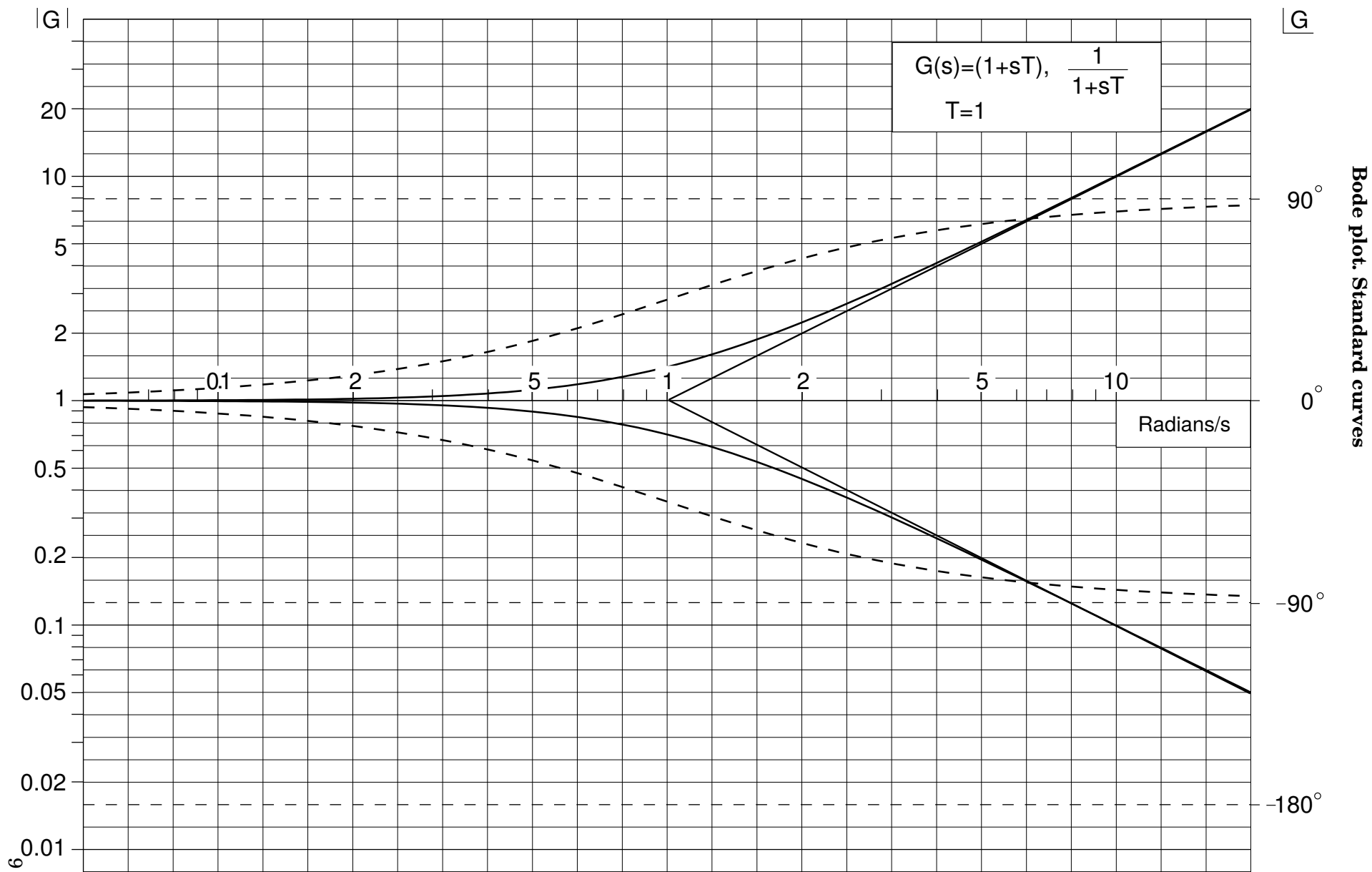


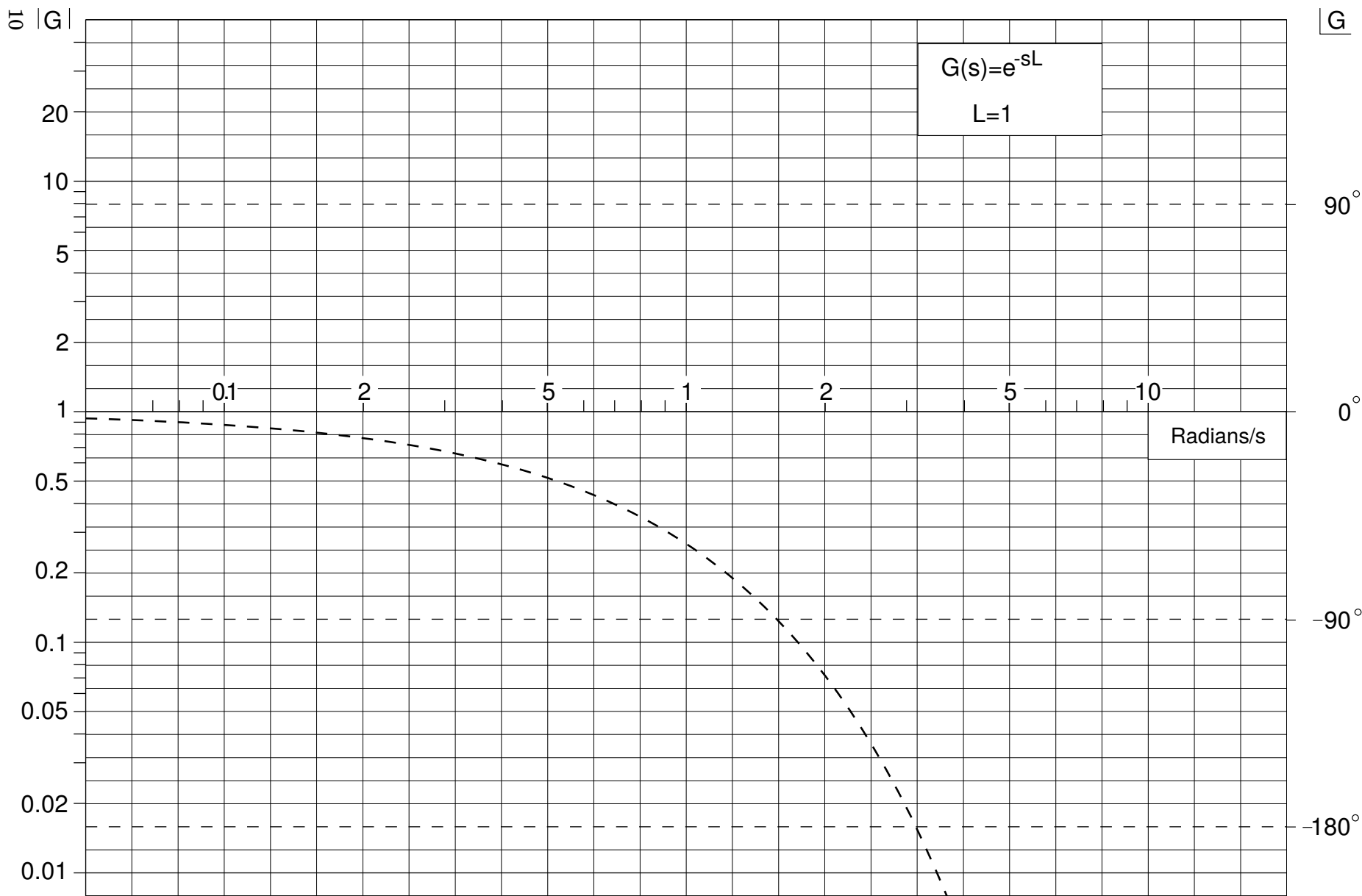
**Transform lexicon, continued**

	Laplace transform $F(s)$	Time function $f(t)$
17	$\frac{s}{(s+a)(s+b)}$	$\frac{ae^{-at} - be^{-bt}}{a-b}$
18	$\frac{a}{(s+b)^2 + a^2}$	$e^{-bt} \sin at$
19	$\frac{s+b}{(s+b)^2 + a^2}$	$e^{-bt} \cos at$
20	$\frac{1}{s^2 + 2\zeta\omega_0s + \omega_0^2}$	$\zeta = 0 \quad \frac{1}{\omega_0} \sin \omega_0 t$ $\zeta < 1 \quad \frac{1}{\omega_0\sqrt{1-\zeta^2}} e^{-\zeta\omega_0 t} \sin(\omega_0\sqrt{1-\zeta^2} t)$ $\zeta = 1 \quad te^{-\omega_0 t}$ $\zeta > 1 \quad \frac{1}{\omega_0\sqrt{\zeta^2-1}} e^{-\zeta\omega_0 t} \sinh(\omega_0\sqrt{\zeta^2-1} t)$
21	$\frac{s}{s^2 + 2\zeta\omega_0s + \omega_0^2}$	$\zeta < 1 \quad \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_0 t} \sin(\omega_0\sqrt{1-\zeta^2} t + \tau)$ $\tau = \arctan \frac{\omega_0\sqrt{1-\zeta^2}}{-\zeta\omega_0}$ $\zeta = 0 \quad \cos \omega_0 t$
	$0 \leq \tau \leq \pi :$	
22	$\frac{a}{(s^2 + a^2)(s+b)}$	$\zeta = 1 \quad (1 - \omega_0 t)e^{-\omega_0 t}$ $\frac{1}{\sqrt{a^2 + b^2}} (\sin(at - \phi) + e^{-bt} \sin \phi)$ $\phi = \arctan \frac{a}{b}$

**Laplace transform table, continued**

	Laplace transform $F(s)$	Time function $f(t)$
23	$\frac{s}{(s^2 + a^2)(s + b)}$	$\frac{1}{\sqrt{a^2 + b^2}} (\cos(at - \phi) - e^{-bt} \cos \phi)$ $\phi = \arctan \frac{a}{b}$
24	$\frac{ab}{s(s + a)(s + b)}$	$1 + \frac{ae^{-bt} - be^{-at}}{b - a}$
25	$\frac{a^2}{s(s + a)^2}$	$1 - (1 + at)e^{-at}$
26	$\frac{a}{s^2(s + a)}$	$t - \frac{1}{a}(1 - e^{-at})$
27	$\frac{1}{(s + a)(s + b)(s + c)}$	$\frac{(b - c)e^{-at} + (c - a)e^{-bt} + (a - b)e^{-ct}}{(b - a)(c - a)(b - c)}$
28	$\frac{\omega_0^2}{s(s^2 + 2\zeta\omega_0s + \omega_0^2)}$	$0 < \zeta < 1$ $1 - \frac{1}{\sqrt{1 - \zeta^2}} e^{-\zeta\omega_0t} \sin(\omega_0\sqrt{1 - \zeta^2}t + \phi)$ $\phi = \arccos \zeta$  $\zeta = 0$ $1 - \cos \omega_0t$
29	$\frac{1}{(s + a)^{n+1}}$	$\frac{1}{n!} t^n e^{-at}$
30	$\frac{s}{(s + a)(s + b)(s + c)}$	$\frac{a(b - c)e^{-at} + b(c - a)e^{-bt} + c(a - b)e^{-ct}}{(b - a)(b - c)(a - c)}$
31	$\frac{as}{(s^2 + a^2)^2}$	$\frac{t}{2} \sin at$
32	$\frac{1}{\sqrt{s}}$	$\frac{1}{\sqrt{\pi t}}$
33	$\frac{1}{\sqrt{s}} F(\sqrt{s})$	$\frac{1}{\sqrt{\pi t}} \int_0^\infty e^{-\sigma^2/4t} f(\sigma) d\sigma$





Bode plot. Standard curves, continued



## Stability

### Asymptotic stability conditions for low-order polynomials

$$\begin{array}{ll} s + a_1 & a_1 > 0 \\ s^2 + a_1s + a_2 & a_1 > 0, \quad a_2 > 0 \\ s^3 + a_1s^2 + a_2s + a_3 & a_1 > 0, \quad a_2 > 0, \quad a_3 > 0, \quad a_1a_2 > a_3 \end{array}$$

### Routh's algorithm

Consider the polynomial

$$F(s) = a_0s^n + b_0s^{n-1} + a_1s^{n-2} + b_1s^{n-3} + \dots$$

Assume that the coefficients  $a_i, b_i$  are real and that  $a_0$  is positive. Form the table

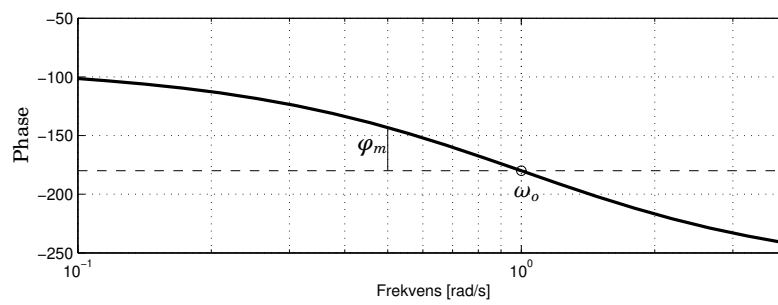
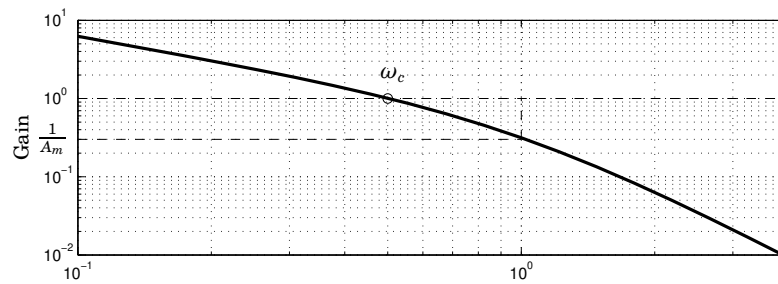
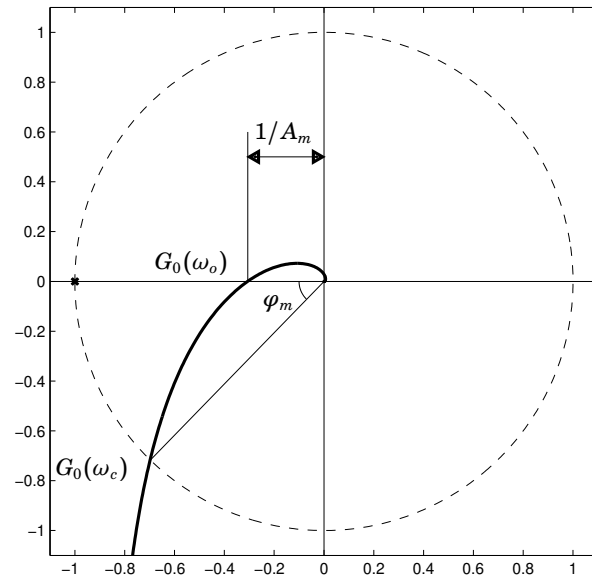
$$\begin{array}{lll} a_0 & a_1 & a_2 \dots \\ b_0 & b_1 & b_2 \dots \\ c_0 & c_1 & c_2 \dots \\ d_0 & d_1 & d_2 \dots \\ \vdots & & \end{array}$$

where

$$\begin{array}{l} c_0 = a_1 - a_0b_1/b_0 \\ c_1 = a_2 - a_0b_2/b_0 \\ \vdots \\ d_0 = b_1 - b_0c_1/c_0 \\ d_1 = b_2 - b_0c_2/c_0 \\ \vdots \end{array}$$

The number of sign changes in the sequence  $a_0, b_0, c_0, d_0 \dots$  equal the number of roots for the polynomial  $F(s)$  in the right half plane  $\text{Re } s > 0$ . All the roots of the polynomial  $F(s)$  lie in the left half plane if all numbers  $a_0, b_0, c_0, d_0, \dots$  are positive.

## Stability margins



**Gain margin:**

$$A_m = 1/|G_0(i\omega_o)|$$

**Phase margin:**

$$\varphi_m = \pi + \arg G_0(i\omega_c)$$

**Delay margin:**

$$L_m = \varphi_m/\omega_c$$

## State feedback and Kalman filtering

### State feedback

If the system

$$\begin{aligned}\frac{dx}{dt} &= Ax + Bu \\ y &= Cx\end{aligned}$$

has the control law

$$u = -Kx + k_r r$$

then the closed-loop system is given by

$$\begin{aligned}\frac{dx}{dt} &= (A - BK)x + Bk_r r \\ y &= Cx\end{aligned}$$

**Criterion for controllability.** The controllable states belong to the linear subspace which is spanned by the columns of the matrix

$$W_s = \begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix}$$

A system is controllable if and only if the matrix  $W_s$  has  $n$  linearly independent columns.

### Kalman filtering

Assume that only the output signal  $y$  can be directly measured. Introduce the model

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + L(y - C\hat{x})$$

The reconstruction error  $\tilde{x} = x - \hat{x}$  satisfies

$$\frac{d\tilde{x}}{dt} = (A - LC)\tilde{x}$$

**Criterion for observability.** The subspace of unobservable states is the null space of the matrix

$$W_o = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}$$

A system is observable if and only if the matrix  $W_o$  has  $n$  linearly independent rows.



## Lead-lag compensation

### Lag compensator

$$G_K(s) = \frac{s+a}{s+a/M} = M \frac{1+s/a}{1+sM/a} \quad M > 1$$

The rule of thumb

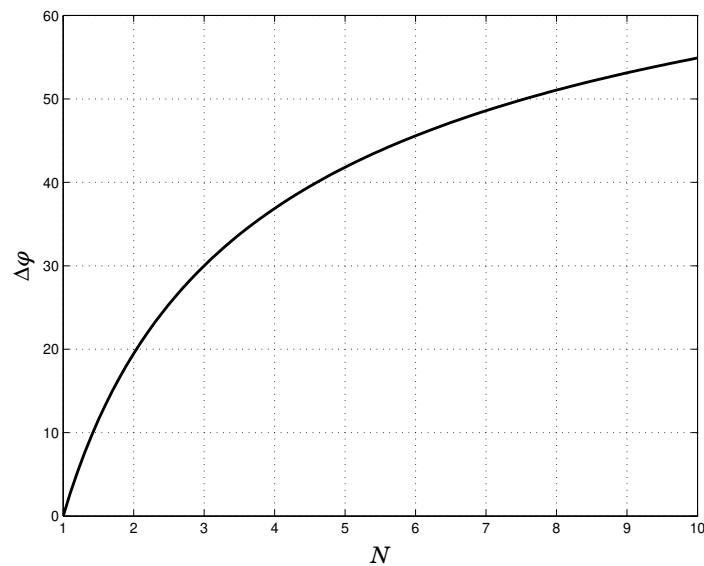
$$a = 0.1\omega_c$$

guarantees that the phase margin is reduced by less than  $6^\circ$ .

### Lead compensator

$$G_K(s) = K_K N \frac{s+b}{s+bN} = K_K \frac{1+s/b}{1+s/(bN)} \quad N > 1$$

The maximum phase advance is given by the figure below:



The peak of the phase curve is located at the frequency

$$\omega = b\sqrt{N}$$

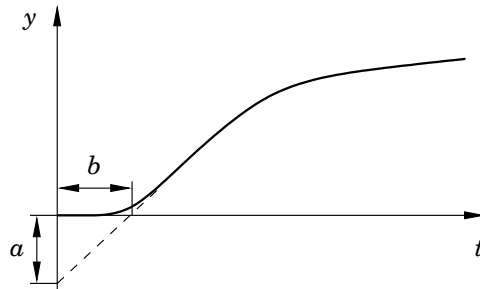
The gain of the compensator at this frequency is

$$K_K\sqrt{N}$$

## Simple PID tuning rules

### The Ziegler-Nichols step response method

Consider the step response for the *open-loop* system. The tangent is drawn from the point on the step response with the maximal slope. From the intersection of the tangent and the coordinate axes the gain  $a$  and time  $b$  are found. The PID-parameters are calculated from the table below.



Controller	$K$	$T_i$	$T_d$
P	$1/a$		
PI	$0.9/a$	$3b$	
PID	$1.2/a$	$2b$	$0.5b$

### The Ziegler-Nichols frequency method

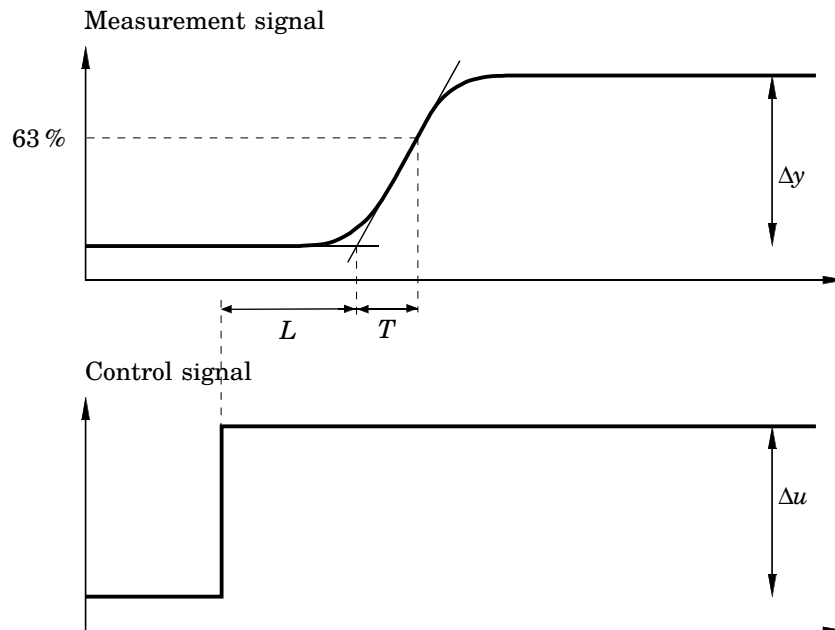
This method is based on observations of the *closed-loop* system. Outline of the procedure:

1. Disconnect the integral and the derivative part of the PID-controller.
2. Adjust  $K$  until the system oscillates with constant amplitude. Denote this value of  $K$  as  $K_0$ .
3. Measure the period  $T_0$  for the oscillation. The different settings for the controller parameters are given in the table below.

Controller	$K$	$T_i$	$T_d$
P	$0.5K_0$		
PI	$0.45K_0$	$T_0/1.2$	
PID	$0.6K_0$	$T_0/2$	$T_0/8$

### The Lambda method

The Lambda method is based on a step response experiment where the static gain  $K_p$ , a deadtime  $L$ , and a time constant  $T$  are determined according to the following figure



where

$$K_p = \frac{\Delta y}{\Delta u}$$

The controller parameters for a PI controller are:

$$K = \frac{1}{K_p} \frac{T}{L + \lambda}$$

$$T_i = T$$

The controller parameters for a PID controller in series and parallel form, respectively, are:

$$K' = \frac{1}{K_p} \frac{T}{L/2 + \lambda} \qquad K = \frac{1}{K_p} \frac{L/2 + T}{L/2 + \lambda}$$

$$T'_i = T \qquad T_i = T + L/2$$

$$T'_d = \frac{L}{2} \qquad T_d = \frac{TL}{L + 2T}$$