Automatic Control – Basic Course

Laboratory Exercise 2

Model construction and calculation of PID Controller

Department of Automatic Control
Lund University
Last updated 2017

1. Introduction

Lab exercise 1 gave practical experience and insight into PID control. However, we lacked systematic methods for choosing the controller parameters. The purpose of this lab is to show how one can construct a mathematical model for the process one wants to control and how to calculate suitable controller settings using this model.

The lab is conducted on the same tank system which was used in lab 1, see figure 1.

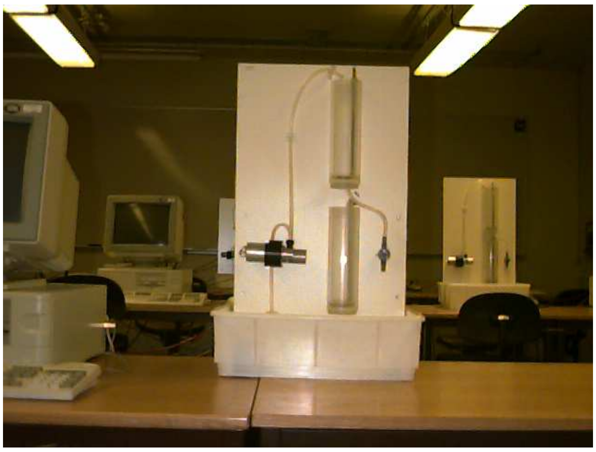


Figure 1 Lab setup.

Preparations

To get out as much as possible of the lab it is important that you master the concepts of linearization, transfer function, characteristic polynomial and pole placement.

You should have read through this lab manual. You should also have worked through the preparatory assignments 2.1, 2.2, 2.3, 2.5, 2.6, 3.1 and 3.5. Cooperation is allowed (and encouraged). Observe that assignments 3.1 and 3.5 are done on an exercise session.

The lab starts with a written test, where two randomly chosen review questions shall be answered. **Both questions must be somewhat correctly answered for you to do the lab. Additionally, you must be able to account for your solutions of the preparatory assignments.** The review questions are found on page 17.

Don't forget to bring the lab manual from lab 1 also to this lab in order to compare your results.

2. Model Construction

In this section we shall deduce a mathematical model for the tank system, starting out with physical principles and construction data. The obtained mathematical model shall then be verified through a couple of experiments.

Assignment 2.1 (Preparation) Give the differential equations which describe how the level in the upper and lower tank, respectively, depend on time. An approximate relation between outflow speed $v(t)$ and tank level $h(t)$ in a tank is given by Toricelli's law:

$$v(t) = \sqrt{2gh(t)}$$

The dynamics in the hoses and the motor can be neglected. Let A_1 and A_2 represent the cross sections of the tanks, a_1 and a_2 the areas of their outflows, respectively. Assume that the flow q from the pump is proportional to the motor voltage u with k being the constant of proportionality.

Assignment 2.2 (Preparation) Show that if the tanks have the same cross section, $A_1 = A_2 = A$, we can write the model as

$$\begin{aligned} \frac{dh_1(t)}{dt} &= -\gamma_1\sqrt{2gh_1(t)} + \beta u(t) \\ \frac{dh_2(t)}{dt} &= \gamma_1\sqrt{2gh_1(t)} - \gamma_2\sqrt{2gh_2(t)} \end{aligned} \tag{1}$$

where $\beta = k/A$, $\gamma_1 = a_1/A$ and $\gamma_2 = a_2/A$.

Calculate theoretical values for the parameters β, γ_1 and γ_2 from the below construction data. Insert your answers into the below table.

The cross sections of the tanks:	$A_1 = A_2 = 2.8 \cdot 10^{-3}$ m^2
The outflow areas of the tanks:	$a_1 = a_2 = 7 \cdot 10^{-6}$ m^2
Constant of proportionality for the pump:	$k = 2.7 \cdot 10^{-6}$ m^3/s/V

Assignment 2.3 (Preparation) In practice all the tank processes do not have exactly the same construction data. Additionally, their properties are changed over time – the holes are sedimented, the pumps are worn, etc. The theoretical parameter values are therefore not always totally reliable. The real values can, however, be estimated through a few simple experiments:

- β can be estimated by blocking the outflow of the upper tank, setting a constant pump voltage and then measuring how long time it takes for the water to rise to a certain level.
- γ_1 and γ_2 can be measured by setting a constant pump voltage, waiting until the system reaches an equilibrium and then reading the stationary levels h_1^0 and h_2^0.

Starting out with equation (1), show how one can calculate experimental values of first β, then γ_1 and finally γ_2 using the above experiments.

Assignment 2.4 Log in according to your lab assistant's instructions. Execute the experiments and calculations according to assignment 2.3 to determine experimental values of β, γ_1 and γ_2. Insert the results in the below table.

	Theoretical values	Experimental values
β		
γ_1		
γ_2		

Check that the experimental values coincide fairly well with the theoretical ones. You should preferably base your controller design on the experimental values.

Assignment 2.5 (Preparation) Linearize the system (1) about an arbitrary equilibrium (h_1^0, h_2^0). (During the lab we will use $h_1^0 = 10$ cm, $h_2^0 = 10$ cm)

Units and Unit Conversions Although unit conversions are in principle simple, they often lead to errors. The problem is especially severe for a control engineer, who often works with many different units within the system. Most often one works with physical units during model construction. Later, when stepping over to control, it is necessary to involve converters and conversion constants.

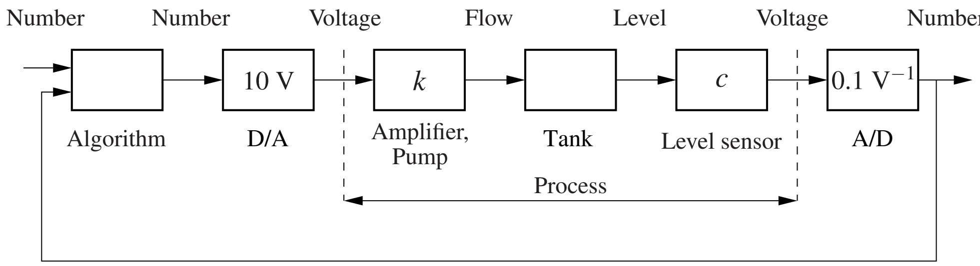


Figure 2 Block diagram for the process with unit converters.

Figure 2 shows a block diagram of the process and controller with all involved converters. It is not obvious where the border between process and controller shall be drawn. It is, however, according to custom to choose the border so that the process inputs and outputs are of same units. With this convention the transfer function of the controller becomes unit-less. In our case the process in- and outputs will be of the units Volt (V). However, note that the control algorithm in the computer works with numbers (i.e. unit-less) because the A/D- and D/A converters contain a conversion factor of 10 V.

Assignment 2.6 (Preparation) Introduce the two measurement signals

$$y_1(t) = c \cdot h_1(t)$$
$$y_2(t) = c \cdot h_2(t)$$

where the level sensors have the proportionality constants $c = 50$ V/m. Show that the linearized system from assignment 2.5 can be described by the following transfer function:

$$\Delta Y_1(s) = \frac{p\tau_1}{1+s\tau_1}\Delta U(s) \tag{2}$$

$$\Delta Y_2(s) = \frac{p\tau_2}{(1+s\tau_1)(1+s\tau_2)}\Delta U(s) \tag{3}$$

Determine the parameters p, τ_1 and τ_2 as functions of the process parameters β, γ_1, γ_2, k, c and the the working point h_1^0, h_2^0.

3. Calculation of Controller Settings

In this section we shall calculate the controller settings for control of the upper and lower tank, respectively. We start out with the mathematical models obtained in the previous section. The controllers will be tested together with the tank system.

Controller parameters will depend on the specifications which we want the closed loop system to fulfill. A specification can have different forms; in this case the poles of the closed loop system shall be given as specification. By suitably placing the poles, one can achieve wanted speed and damping of the closed loop system.

In this lab we will work with PI- and PID-controllers. By choosing the PI(D)-parameters suitably, we can obtain a pre-specified characteristic polynomial (transfer function denominator polynomial) for the closed loop system, see figure 3.

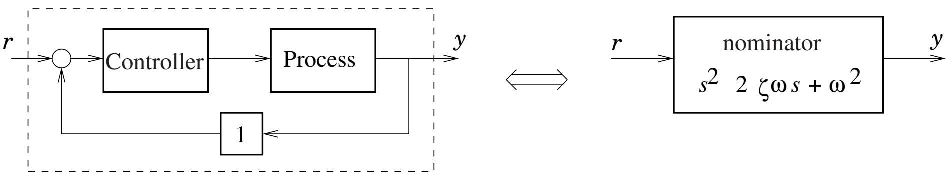


Figure 3 The closed loop system is specified by a desired characteristic polynomial.

Control of the Upper Tank

Assignment 3.1 (Preparation) Use the model (2) from assignment 2.6 to design a PI-controller,

$$u(t) = K\left(e(t) + \frac{1}{T_i}\int_0^t e(\tau)d\tau\right) \quad \Leftrightarrow \quad U(s) = K\left(1+\frac{1}{sT_i}\right)E(s)$$

for control of the upper tank. Choose the controller parameters such that the closed loop system gets a relative damping ζ and an undamped natural frequency ω, i.e. such that the closed loop system gets a characteristic polynomial of the form

$$s^2 + 2\zeta\omega s + \omega^2$$

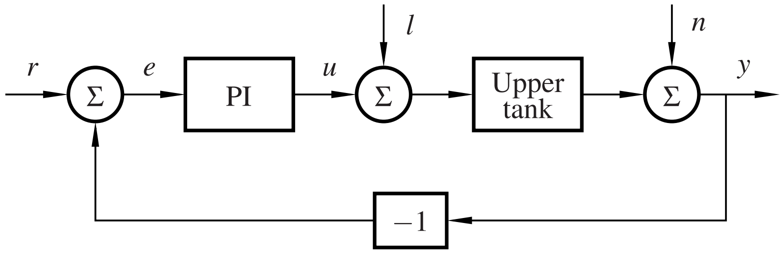


Figure 4 Block diagram for control of the upper tank.

In the answer K and T_i shall be expressed in the process parameters p and τ_1 together with the design parameters ω and ζ.

Poles and Zeros We shall begin by investigating control of the upper tank. A block diagram of the closed loop system is shown in figure 4. It is marked in the block diagram where load disturbances, l, and measurement noise, n, enter. The transfer functions from reference to output (G_{yr}), from load disturbances to output (G_{yl}) and from measurement noise to output (G_{yn}) are given below.

$$G_{yr} = \frac{pK(s+\frac{1}{T_i})}{s^2+s(\frac{1}{\tau_1}+pK)+\frac{pK}{T_i}}$$
$$G_{yl} = \frac{sp}{s^2+s(\frac{1}{\tau_1}+pK)+\frac{pK}{T_i}}$$
$$G_{yn} = \frac{s(s+\frac{1}{\tau_1})}{s^2+s(\frac{1}{\tau_1}+pK)+\frac{pK}{T_i}}$$

A step load disturbance l corresponds to opening the side valve of the upper tank. The measurement disturbances n can model measurement noise or a constant offset error in the level sensor of the tank.

The three transfer functions have the same denominator polynomial, whereas their nominator polynomials differ. As the controller parameters are changed, the poles of the system will move. In the transfer function from reference value to output, G_{yr}, also the zeros of the system will move. The zeros of systems G_{yl} and G_{yn} are unaffected by the controller parameters. If we want to determine how the pole placement affects the system, we shall mainly study the response to load disturbances. If we want to see the combined effect of poles and zeros we shall study the the response to a change in reference value.

Assignment 3.2 Fix ζ to 1 and vary ω according to the below table. Assume that the stationary level is $h_1^0 = 10$ cm and calculate the parameters K and T_i of a PI-controller, for every value of ω. This can be done using the Octave[1] script `calcpi` according to the following example (insert your estimated values of `beta`, `gamma1` and `gamma2`):

```
>> beta = ... ;
>> gamma1 = ... ;
```

[1] Octave is a free MATLAB clone.

```
>> gamma2 = ... ;
>> omega = 0.1;
>> zeta = 1;
>> calcpi
K =
    3.7861
Ti =
   18.249
```

Also view the script by typing

```
>> type calcpi
```

and compare the calculations with your preparatory assignments.

Try the controllers on the upper tank and investigate the responses to reference value changes and load disturbances. Draw the responses in the below time diagrams (cf. figure 5). Also insert the location of the poles in the pole-zero plots and compare with the shape of the responses; especially observe their speed.

Carry out the experiments as follows:

1. Make sure that the interface is set to **PI**-control of the **upper** tank.
2. Make sure that the side valve of the upper tank is closed.
3. Enter the controller parameters K and T_i.
4. Enter the reference value 6 cm ($r = 0.3$) and wait until all signals have become stationary.
5. Issue a reference value change to 10 cm ($r = 0.5$) and draw its response. Enter the rise time and the size of the overshoot (see figure 5) in the table and also wether the control signal saturates (i.e. reaches its max value) and for how long.
6. When the system has anew reached a stationary state, open the side valve and draw the response to the load disturbance. Enter the settling time for the load disturbance in the table.

Finally, also try the controller for the upper tank which you ended up with in lab 1. (Fill out the last row of the table.)

				Change in reference value			Load disturbance
ω	ζ	K	T_i	Rise time	Overshoot	Saturation	Settling time
0.1	1						
0.2	1						
0.5	1						

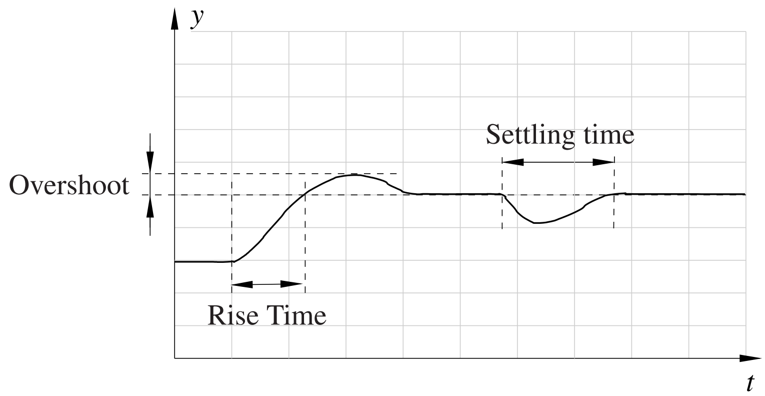
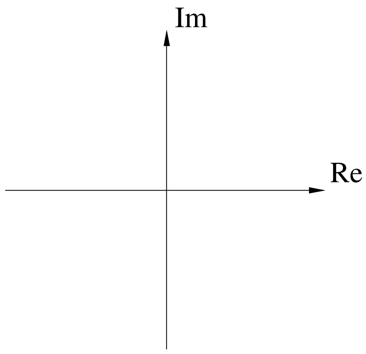
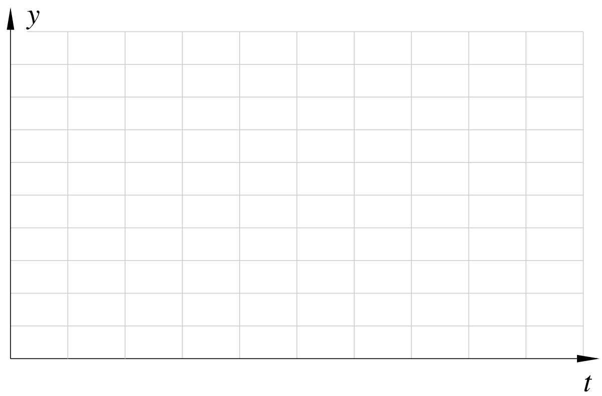
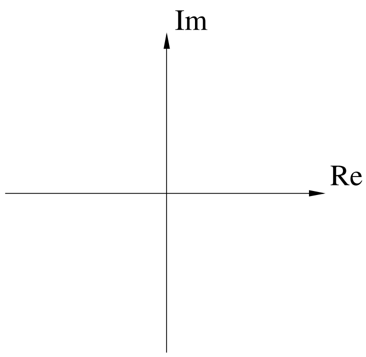
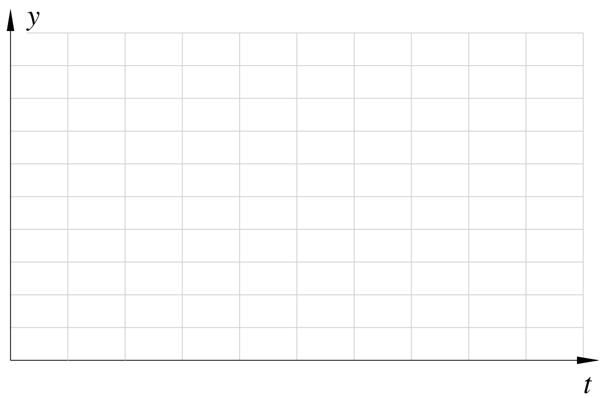


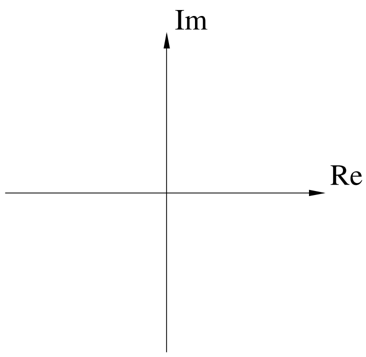
Figure 5 The definition of rise time and overshoot when changing reference value and settling time of a load disturbance.

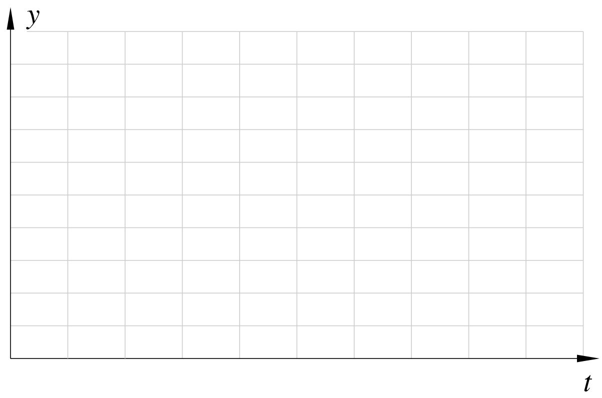


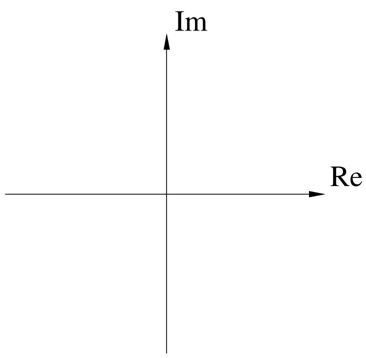








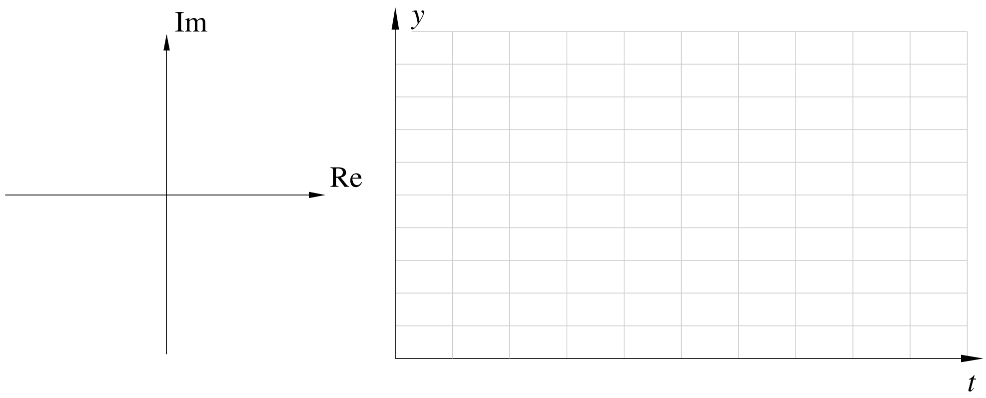


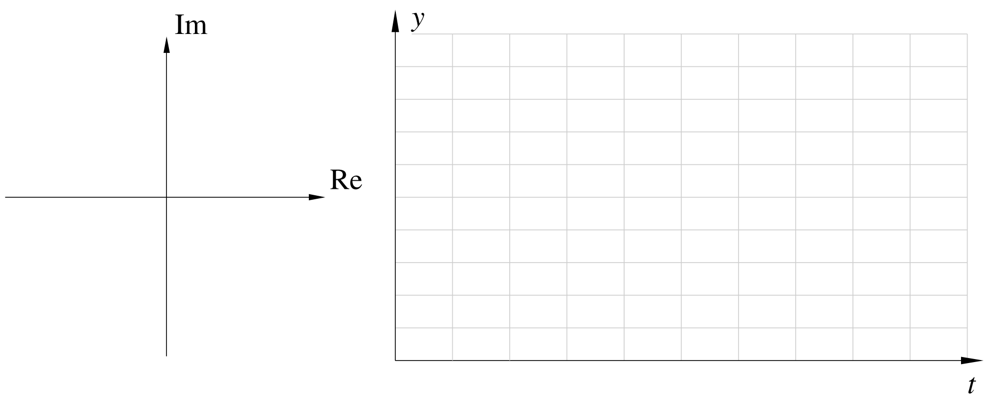


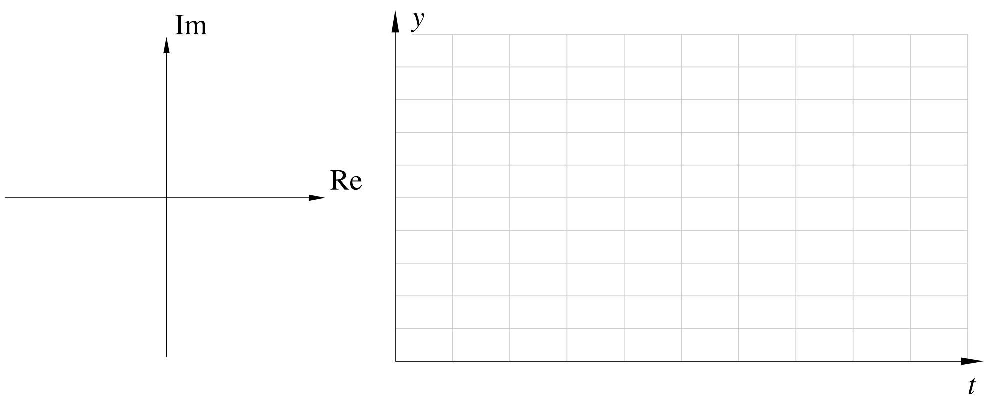
Assignment 3.3 Now fix ω to 0.2 and instead vary ζ according to the below table. Calculate the controller parameters K and T_i using Octave in the same way as previously. Try the controllers on the upper tank and investigate the responses to changes in reference value and load disturbances. Draw the responses in the time diagrams below. Also draw the locations of the poles in the pole-zero plots and compare with the shape of the responses; especially observe their damping.

Carry out the experiments in the same way as in the previous assignment.

				Change in reference value			Load disturbance
ω	ζ	K	T_i	Rise time	Overshoot	Saturation	Settling time
0.2	0.7						
0.2	0.4						
0.2	0.1						







Assignment 3.4 (Extra) Use one of the controllers calculated in assignment 3.3. Decrease the gain K to one tenth of its calculated value. How will the step response change? Use the controller and try to explain the result.

$$K_{old} =$$
$$K_{new} =$$

Hint: From assignment 3.1 we can obtain the relation

$$\omega = \sqrt{\frac{Kp}{T_i}}$$
$$\zeta = \frac{Kp + \frac{1}{\tau_1}}{2\omega} \approx \frac{\sqrt{KpT_i}}{2}$$

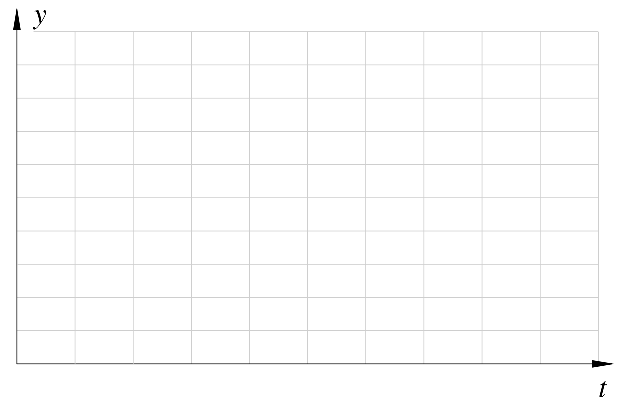


Figure 6 Step response for K_{old} and K_{new}.

Control of the Lower Tank

Assignment 3.5 (Preparation) Use the model (3) from assignment 2.6 in order to design a PID-controller,

$$u(t) = K\left(e(t) + \frac{1}{T_i}\int_0^t e(\tau)d\tau + T_d\frac{de(t)}{dt}\right) \quad \Leftrightarrow \quad U(s) = K\left(1 + \frac{1}{sT_i} + sT_d\right)E(s)$$

to control the level in the lower tank. Choose controller parameters such that the closed loop system gets the characteristic polynomial

$$(s+\alpha\omega)(s^2+2\zeta\omega s+\omega^2)$$

In the answer K, T_i and T_d shall be given in the process parameters p, τ_1 and τ_2 together with the design parameters ω, ζ and α.

Poles and Zeros We shall now investigate control of the lower tank. A block diagram of the closed loop system is shown in figure 7. It is marked in the block diagram where load disturbances, l_1, l_2, and measurement noise, n, can enter. The transfer functions from reference to output (G_{yr}), from load disturbances to output (G_{yl_1}, G_{yl_2}) and from measurement noise to output (G_{yn}) are given below.

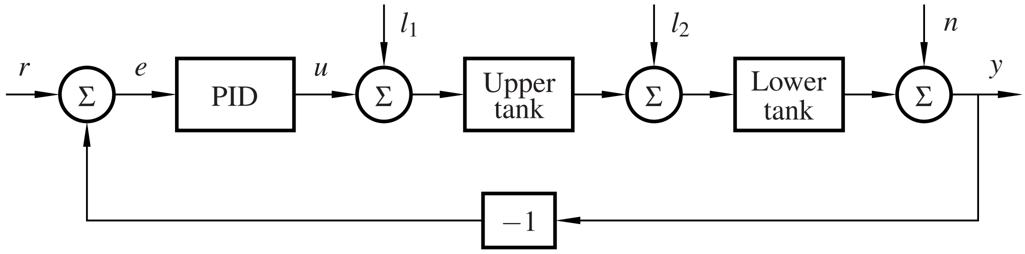


Figure 7 Block diagram for control of the lower tank.

$$
\begin{aligned}
G_{yr} &= \frac{Kp(s^2\frac{T_d}{\tau_1} + s\frac{1}{\tau_1} + \frac{1}{T_i\tau_1})}{s^3 + s^2(\frac{1}{\tau_1} + \frac{1}{\tau_2} + \frac{pKT_d}{\tau_1}) + s(\frac{1}{\tau_1\tau_2} + \frac{pK}{\tau_1}) + \frac{pK}{T_i\tau_1}} \\
G_{yl_1} &= \frac{s\frac{p}{\tau_1}}{s^3 + s^2(\frac{1}{\tau_1} + \frac{1}{\tau_2} + \frac{pKT_d}{\tau_1}) + s(\frac{1}{\tau_1\tau_2} + \frac{pK}{\tau_1}) + \frac{pK}{T_i\tau_1}} \\
G_{yl_2} &= \frac{s\frac{1}{\tau_1}(s + \frac{1}{\tau_1})}{s^3 + s^2(\frac{1}{\tau_1} + \frac{1}{\tau_2} + \frac{pKT_d}{\tau_1}) + s(\frac{1}{\tau_1\tau_2} + \frac{pK}{\tau_1}) + \frac{pK}{T_i\tau_1}} \\
G_{yr} &= \frac{s(\frac{1}{\tau_1\tau_2} + s(\frac{1}{\tau_2} + \frac{1}{\tau_1}) + s^2)}{s^3 + s^2(\frac{1}{\tau_1} + \frac{1}{\tau_2} + \frac{pKT_d}{\tau_1}) + s(\frac{1}{\tau_1\tau_2} + \frac{pK}{\tau_1}) + \frac{pK}{T_i\tau_1}}
\end{aligned}
$$

A step load disturbance l_1 corresponds to opening the side valve. The load disturbance l_2 corresponds to an extra inflow to the lower tank, whereas measurement disturbances n can model measurement noise or constant measurement errors in the level sensor of the tank.

The four transfer functions have the same denominator polynomial, whereas the nominator polynomials differ. As the controller parameters are changed, the poles of the system will move. In the transfer function from reference value to output, G_{yr}, the zeros of the system will also move. The zeros of the systems G_{yl_1}, G_{yl_2} and G_{yn} are unaffected by the controller parameters. If we wish to investigate how the location of the poles influence the system, we shall mainly study the response to load disturbances. If we want to investigate the combined effect of poles and zeros, we may study the response to changes in reference value.

Assignment 3.6 Fix ζ to 0.7, α to 1 and vary ω according to the below table. Assume that the stationary level is $h_2^0 = 10$ cm and calculate the parameters for a PID -controller using Octave and the script `calcpid` as below:

```
>> omega = 0.04;
>> zeta = 0.7;
>> alpha = 1;
>> calcpid
K =
    4.1869
Ti =
   55.210
Td =
   17.259
```

Also view the script by typing

```
>> type calcpid
```

and compare the calculations with the ones in your preparation assignments.

Try the controller on the lower tank and investigate the response to changes in reference value and load disturbances. Draw the responses in the below time diagrams. Also enter the locations of the poles in the pole-zero plots and compare with the properties of the responses; especially observe their speed.

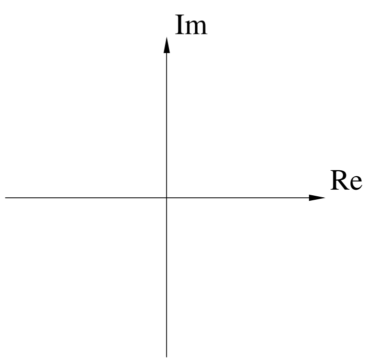
Carry out the experiments as follows

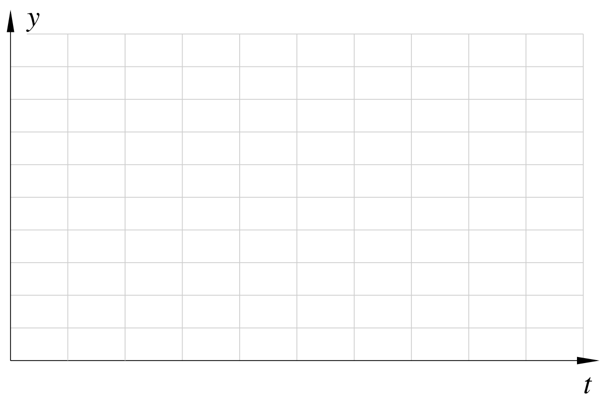
1. Make sure that the interface is set to **PID**-control of the **lower** tank.
2. Make sure that the side valve of the upper tank is closed.
3. Set the controller parameters K, T_i and T_d.
4. Set the reference value at 6 cm ($r = 0.3$) and wait until all signals have become stationary. (The ”Optimal” button could be used for fast reset.)
5. Issue a change in reference value to 10 cm ($r = 0.5$) and draw its response. Enter the rise time and overshoot corresponding in the table. Also write down wether the control signal saturates, and for how long.
6. When the system is anew stationary, open the side valve and draw the response to the load disturbance. Enter the settling time for the load disturbance in the table.

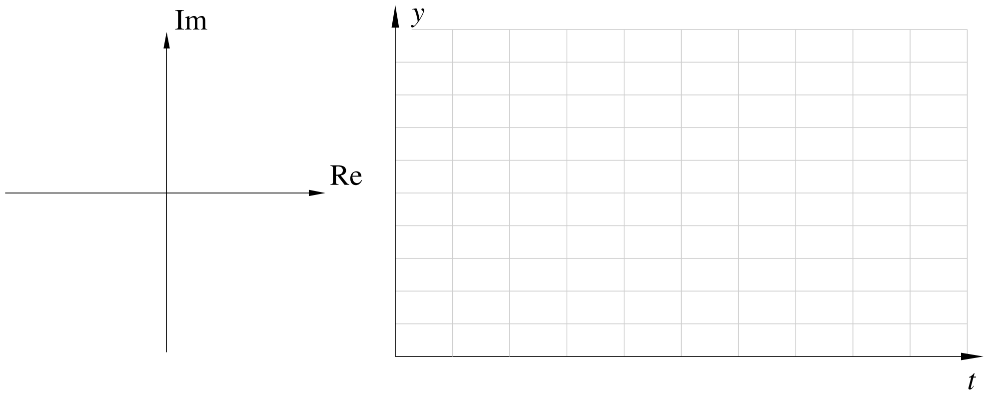
Finally, try the controller for the lower tank which you ended up with in lab 1. (Fill out the last row of the table.)

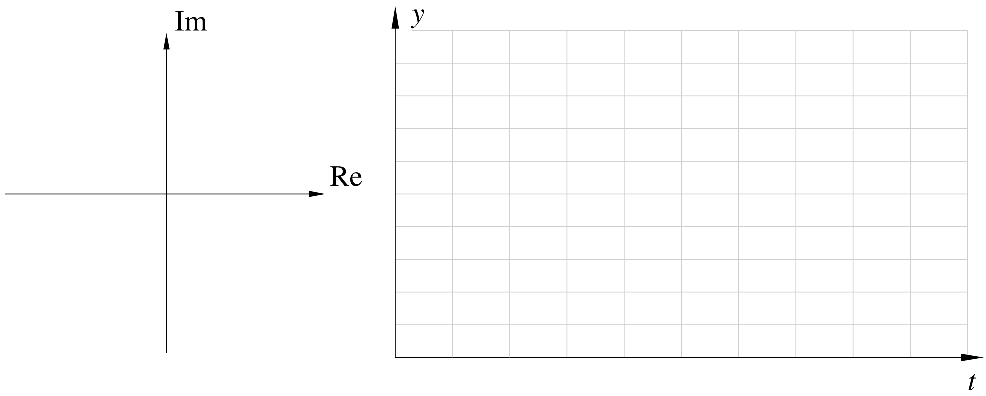
N.B.! These experiments take quite some time to perform. Preferably work with the summary in chapter 4 during that time.

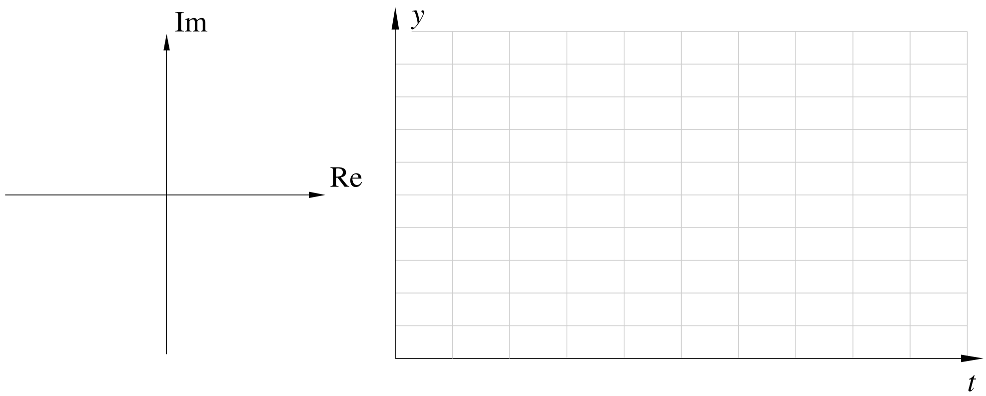
						Change in reference value			Load disturbance
ω	ζ	α	K	T_i	T_d	Rise time	Overshoot	Saturation	Settling time
0.035	0.7	1							
0.05	0.7	1							
0.1	0.7	1							









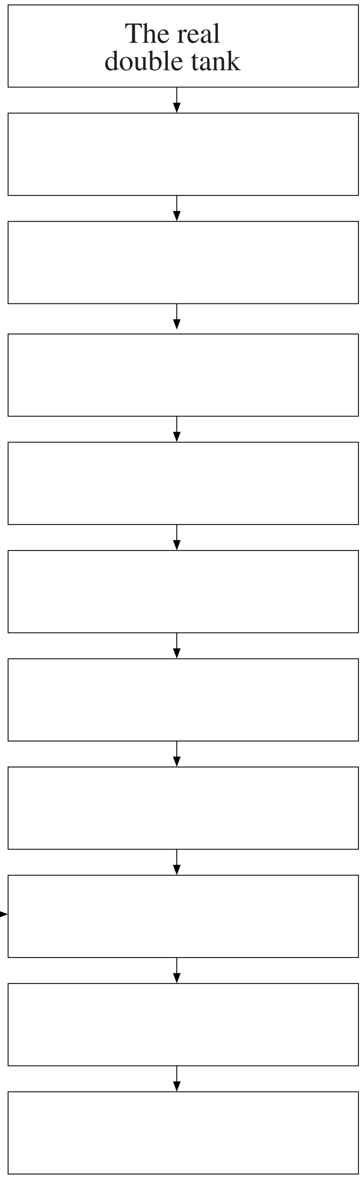


4. Summary

This summary intends to illustrate the workflow used in controller design and to post relevant questions which you shall be able to answer after finishing the experiments. The lab assistant will go through your summary before you pass this lab.

Assignment 4.1 Enter the stages you have gone through before and during the lab in the empty boxes of the below figure, in correct order. (Observe that the parameter estimation experiments are excluded. Where would they fit in?)

- Closed loop transfer function $Y(s) = \frac{G_p G_r}{1+G_p G_r} R(s)$
- Physical modeling
- Nonlinear differential equation
  $\dot{x} = f(x,u)$
- Linear differential equation
  $\dot{x} = ax + bu$
- Linearization
- Laplace transform
- Specification as pole placement
  $s^2 + 2\zeta\omega s + \omega^2 = 0$
- Test on process
- Evaluation
- Expressions for controller parameters
  $K = \dots,\ T_i = \dots,$
- Transfer functions
  $Y(s) = G(s)U(s)$



Assignment 4.2 Give at least two limitations of the real process which are not captured by the mathematical model (1).

Assignment 4.2 During-PI control of the upper tank, how are the poles of the closed loop system changed when the parameter ω is increased? How does this affect responses to changes in reference and load disturbances?

How do K and T_i change when ω is increased? Why don't we try $\omega = 5$ rad/s?

Assignment 4.3 During PI control of the upper tank, how are the poles of the closed loop system changed if the parameter ζ is decreased? How does it affect the responses to changes in reference value and load disturbances, respectively? How would the step response look in case we chose $\zeta = 0$?

Assignment 4.4 Why don't we use the D-part when controlling the upper tank?

Assignment 4.5 During PID-control of the lower tank, how many poles does the closed loop system have?

How are the poles of the closed loop system changed if the parameter ω is increased? What effect does this have to the responses to changes in reference value and load disturbances, respectively?

How is K, T_i and T_d changed when ω is increased? Why don't we try $\omega = 1$ rad/s?

Assignment 4.7 Enter your recommendations for suitable controller parameters in the below table. Compare with the parameters you ended up with in lab 1.

	Upper tank	Lower tank
P	K =	K =
PI	K = T_i =	K = T_i =
PID	K = T_i = T_d =	K= T_i = T_d =

Review Questions for Lab 2

1. Determine all stationary points (x^0, u^0, y^0) for the system

$$\frac{dx}{dt} = -a\sqrt{x} + bu$$
$$y = cx$$

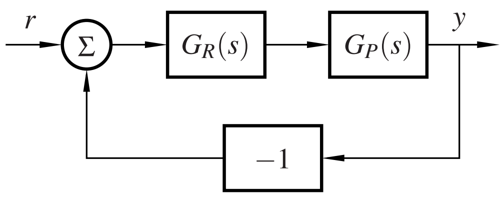
2. Linearize the system

$$\frac{dx}{dt} = -a\sqrt{x} + bu$$
$$y = cx$$

   about the stationary point (x^0, u^0, y^0).

3. Write down the transfer function for a
   (a) P-controller
   (b) PI-controller
   (c) PID-controller

4. Determine the closed loop transfer function for the open loop



   when $G_R(s) = K$ and $G_P(s) = \frac{1}{1+sT}$

5. In second order systems it is common to talk about two parameters

ζ	(relative damping)
ω	(natural frequency)

   Illustrate how these parameters define the location of the poles in a pole zero plot.

6. The transfer function of a system can be written as

$$G(s) = K\frac{Q(s)}{P(s)}$$

   Observe the pole-zero plot of the system (figure to the right) and determine $Q(s)$ and $P(s)$, respectively.