LUND UNIVERSITY

Department of
AUTOMATIC CONTROL

FRTN 15 Predictive Control

Final Exam March 14, 2017, 8am - 13pm

General Instructions

This is an open book exam. You may use any book you want, including the slides from the lecture, but no exercises, exams, or solution manuals are allowed. Solutions and answers to the problems should be well motivated. The exam consists of 6 problems. The credit for each problem is indicated in the problem. The total number of credits is 25 points. Preliminary grade limits:

Grade 3:	12 – 16 points
Grade 4:	17 – 21 points
Grade 5:	22 – 25 points

Results

The results of the exam will be presented in LADOK by March 24.

1. Choose suitable controller designs for the following control problems (a-c). Provide clear and concise justification for your design choices along with necessary adjustments according to the stated specifications. No calculations are needed.

a. Design a controller for the MIMO system

$$\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \end{pmatrix} = A \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} + B \begin{pmatrix} u_1(k) \\ u_2(k) \end{pmatrix},$$
$$\begin{pmatrix} y_1(k) \\ y_2(k) \end{pmatrix} = C \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix}$$

so that the constraints

$$\begin{pmatrix} l_{x_1} \\ l_{x_2} \end{pmatrix} \leq \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} \leq \begin{pmatrix} d_{x_1} \\ d_{x_2} \end{pmatrix}, \quad \begin{pmatrix} l_{u_1} \\ l_{u_2} \end{pmatrix} \leq \begin{pmatrix} u_1(k) \\ u_2(k) \end{pmatrix} \leq \begin{pmatrix} d_{u_1} \\ d_{u_2} \end{pmatrix}, \quad \forall k$$

are fulfilled. The controller should provide zero offset.

The controller should also account for that variations in u_2 are much more expensive than variations in u_1 and that the control performance of y_1 is of higher priority than that of y_2. (2 p)

b. Design a controller for the system

$$y_{k+2} = -a_1 y_{k+1} - a_2 y_k + b_0 u_{k+1} + b_1 u_k$$

so that the closed-loop dynamics from set-point u^c to output y are given by

$$y_{k+2} = -a_{m1} y_{k+1} - a_{m2} y_k + b_{m0} u^c_{k+1} + b_{m1} u^c_k$$

The system parameters a_1, a_2, b_0 and b_1 are slowly time varying and unknown, it is also expected that the system is non-minimum phase.
The controller should have integral action. (2 p)

c. Design a controller for the system

$$y_k = H(q) u_k$$

where y should follow a known and repetitive signal u_c in a manufacturing application. The system $H(q)$ is not easily modeled but is known to be stable. (1 p)

Solution problemsonly

a.
- MPC is a suitable control approach for systems with constraints since the constraints are included in the optimization problem solved by the MPC.
- In order to obtain zero offset, disturbance states acting on the outputs could be introduced, these states can then be estimated using a Kalman filter.

- Due to the higher cost of using u_2, the control-action weight matrix R should be chosen so that $R(1,1) << R(2,2)$.
- Due to the importance of controlling y_1, the output weight matrix Q should be chosen so that $Q(1,1) >> Q(2,2)$.

b.
- A direct or indirect self-tuning regulator is suitable for this type of control problem. Here we adapt the indirect approach.
- The controller is on the form $R(q)u_k = S(q)y_k + T(q)u_k^c$ where R, S and T depends on H and the closed-loop specifications.
- The model parameters are estimated using the RLS algorithm with a forgetting factor λ since the parameters are time-varying, λ can be set close to 1 since the parameter variation is known to be slow.
- With the estimated model parameters the closed-loop poles are placed according to the closed-loop specifications. This is done by solving a Diophantine equation

$$AR + BS = A_o A_m B^+$$

which gives expressions for R and S. T is obtained from the equation

$$\frac{BT}{AR + BS} = \frac{B_m}{A_o A_m B^+}$$

In order to fulfill the specifications we have to include a factor $q-1$ in the controller denominator R. Since B is non-minumum phase it not possible to cancel B, therefore B has to be included in B_m. With these specifications we have to introduce an observer polynomial A_o of second order.

c. Iterative Learning Control is suitable since the tracking problem is repetetive. After each sequence of u_c, the input sequence $u_{k+1}(t)$ is updated accordingly

$$u_{k+1}(t) = Q(q)(u_{k+1}(t) + L(q)(u_c(t) - y_k(t)))$$

where Q is a low-pass filter and $L(q)$ is a corrective filter. The filters have to be chosen so that the ILC iterations are stable.

2.

a. Compute the controller parameters for an indirect self-tuning regulator with respect to the system

$$y_k = H(q)u_k + w_k$$

where w is unit-variance white Gaussian noise and

$$H(q) = \frac{b_0 q + b_1}{q^2 + a_1 q + a_2}$$

so that the transfer function from set-point u_c to output y is given by

$$H_m(q) = \frac{b_{m0} q + b_{m1}}{q^2 + a_{m1} q + a_{m2}}$$

The model parameters are unknown and constant. The zero of the system H should be cancelled in the design and the controller should have integral action. (2 p)

b. Modify the indirect self-tuning regulator so that it becomes direct. This time the controller should not have integral action. (2 p)

c. It was found that the pole-cancellation led to oscillating u. Suggest a minor modification of the direct self-tuning regulator that solves the problem. (1 p)

Solution

a. First we have to estimate the model parameters, with an indirect STR, this is done by applying the RLS algorithm w.r.t. the linear-regression model

$$y_k = \phi_k^T \theta + w_k$$

where

$$\begin{aligned} y_k &= y_k \\ \theta &= \begin{pmatrix} a_1 & a_2 & b_0 & b_1 \end{pmatrix} \\ \phi_k &= \begin{pmatrix} -y_{k-1} & -y_{k-2} & u_{k-1} & u_{k-2} \end{pmatrix} \end{aligned}$$

A forgetting factor is not needed since the parameters are known to be constant.

The controller is given by

$$Ru = -Sy + Tu_c$$

which gives the closed-loop transfer function

$$y = \frac{BT}{AR + BS} u_c$$

Pole placement with zero cancellation gives the equation

$$A(q-1)R' + B^- S = A_o A_m$$

where $R = B^+(q-1)R'$, $B^- = b_0$ and $B^+ = q + b_1/b_0$, $q - 1$ is introduced to obtain integral action.
The admissibility condition gives that S should be of second order in order to fulfill

$$\deg(S) < \deg(A) + 1$$

This gives $R = B^+(q-1)$ since $\deg(R) = \deg(S) = \deg(T)$ and an observer polynomial $A_o = a_o + z$ of first order.

Now, the equation is given by

$$(q^2 + a_1 q + a_2)(q-1) + b_0(s_0 q^2 + s_1 q + s_2) = (q + a_o)(q^2 + a_{m1} q + a_{m2})$$

from which we obtain the coefficients in S

$$\begin{aligned} a_1 - 1 + b_0 s_0 &= a_o + a_{m1} \\ a_2 - a_1 + b_0 s_1 &= a_{m1} a_o + a_{m2} \\ -a_2 + b_0 s_2 &= a_{m2} a_o \ \Rightarrow \end{aligned}$$

$$
\begin{aligned}
s_0 &= (a_o + a_{m1} - a_1 + 1)/b_0 \\
s_1 &= (a_{m1}a_o + a_{m2} - a_2 + a_1)/b_0 \\
s_2 &= (a_{m2}a_o + a_2)/b_0
\end{aligned}
$$

This gives the controller

$$
\begin{aligned}
R &= B^+(q-1) \\
S &= s_0 z^2 + s_1 z + s_2 \\
T &= B_m A_o / b_0
\end{aligned}
$$

where T is obtained by matching

$$\frac{BT}{AR+BS} = \frac{B_m}{A_m}$$

b. In direct MRAC we want to formulate a regression problem from which we can estimate the controller parameters directly. This can be achieved by letting the Diophantine equation

$$AR + B^- S = A_o A_m$$

operate on y. This gives

$$
\begin{aligned}
AR'y(t) + B^- Sy(t) &= A_o A_m y(t) \Rightarrow \\
BR'u(t) + B^- Sy(t) &= A_o A_m y(t) \Rightarrow \\
B^- Ru(t) + B^- Sy(t) &= A_o A_m y(t) \Rightarrow
\end{aligned}
$$

$$y(t) = \tilde{R} u_f(t - d_0) + \tilde{S} u_f(t - d_0) \tag{1}$$

where

$$y_f(t) = \frac{1}{A_o^*(q^{-1}) A_m^*(q^{-1})} y(t), \;\; u_f(t) = \frac{1}{A_o^*(q^{-1}) A_m^*(q^{-1})} u(t)$$

$\tilde{R} = b_0 R$, $\tilde{S} = b_0 S$ and d_0 is the pole excess of H. This is now a linear regression problem from which R and S can be estimated. T is computed in the same way as before. This time $\deg(R) = \deg(S) = \deg(T) = 1$ and $\deg(A_o) = 0$ since we no longer demand integral action.

c. Zero cancellation can be avoided by increasing d_0 from 1 to 2 in Eq (1), see lecture notes p. 167.

3.

a. Design a mimimum-variance 2-step ahead predictor for the system

$$y_{k+2} = 1.5 y_{k+1} - 0.5 y_k + 2 w_{k+2} - 2.4 w_{k+1} + 1.2 w_k$$

and compute its prediction-error variance. Here w is unit-variance white Gaussian noise. (2 p)

b. Which of the following statements concerning minimum-variance control (MVC) are true/false, motivate your answer:

1. The MVC can be thought of as consisting of two parts: one predictor which predicts the effect of the disturbance of the output, and one dead-beat regulator which computes the control signal required to make the predicted output equal to the desired value.
2. MVC is good at handling poorly damped zeros.
3. MVC is insensitive to parameter variations.

(1.5 p)

Solution

a. The Diophantine equation to be solved is given by

$$C^*(q^{-1}) = A^*(q^{-1})F^*(q^{-1}) + q^{-2}G^*(q^{-1})$$

where F and G is of order 1.

$$\begin{aligned}2 - 2.4q^{-1} + 1.2q^{-2} &= (1 - 1.5q^{-1} + 0.5q^{-2})(f_0 + f_1q^{-1}) + q^{-2}(g_0 + g_1q^{-1}) \\ &= f_0 + (f_1 - 1.5f_0)q^{-1} + (-1.5f_1 + 0.5f_0 + g_0)q^{-2} + (g_1 + 0.5f_1)q^{-3}\end{aligned}$$

Matching the coefficients gives the system of equations

$$\begin{aligned}2 &= f_0 \\ -2.4 &= f_1 - 1.5f_0 \\ 1.2 &= -1.5f_1 + 0.5f_0 + g_0 \\ 0 &= g_1 + 0.5f_1\end{aligned}$$

with the solution

$$\begin{aligned}f_0 &= 2 \\ f_1 &= 0.6 \\ g_0 &= 1.1 \\ g_1 &= -0.3\end{aligned}$$

This gives the predictor

$$\hat{y}_{k+2} = \frac{G^*(q^{-1})}{C^*(q^{-1})}y_k$$

with the estimation-error variance

$$E\{(\hat{y}_{k+2} - y_{k+2})^2\} = (f_0^2 + f_1^2)\sigma_w^2 = 4.36$$

b. 1. True, in MVC we have that

$$\frac{G^*(q^{-1})}{C^*(q^{-1})}y_k + \frac{B^*(q^{-1})F^*(q^{-1})}{C^*(q^{-1})}u_k = 0$$

the left hand side of this expression is the optimal $d - step$ ahead prediction of y_k, u_k is then chosen so that this prediction is 0.

2. MVC introduces zero cancellation and the system zero polynomial B ends up as poles in the controller. Poorly damped zeros thus lead to undamped control signals.
3. It is well-known that an optimal solution under special circumstances may be sensitive to parameter variations. A good example is the MVC controller in Lab 2, in this case sensitivity was reduced by using an adaptive MVC controller.

4. Fig. 1 presents the estimation performance of an RLS-algorithm for different initial covariance matrices P_0 and different input sequences u for the system

$$y_k = ay_{k-1} + bu_{k-1} + w_k, \;\; w_k \sim N(0, \sigma_w^2)$$

Match the RLS experiments in Fig. 1 with the following RLS settings

1. u - square wave with amplitude 2 and period 10, $\sigma_w^2 = 0.5, \;\; P_0 = 0.05I$.
2. $u = 1$, $\sigma_w^2 = 0.5, \;\; P_0 = 1000I$.
3. $u = 2\sin(t)$, $\sigma_w^2 = 25, \;\; P_0 = 1000I$.
4. u - square wave with amplitude 2 and period 10, $\sigma_w^2 = 0.5, \;\; P_0 = 1000I$.

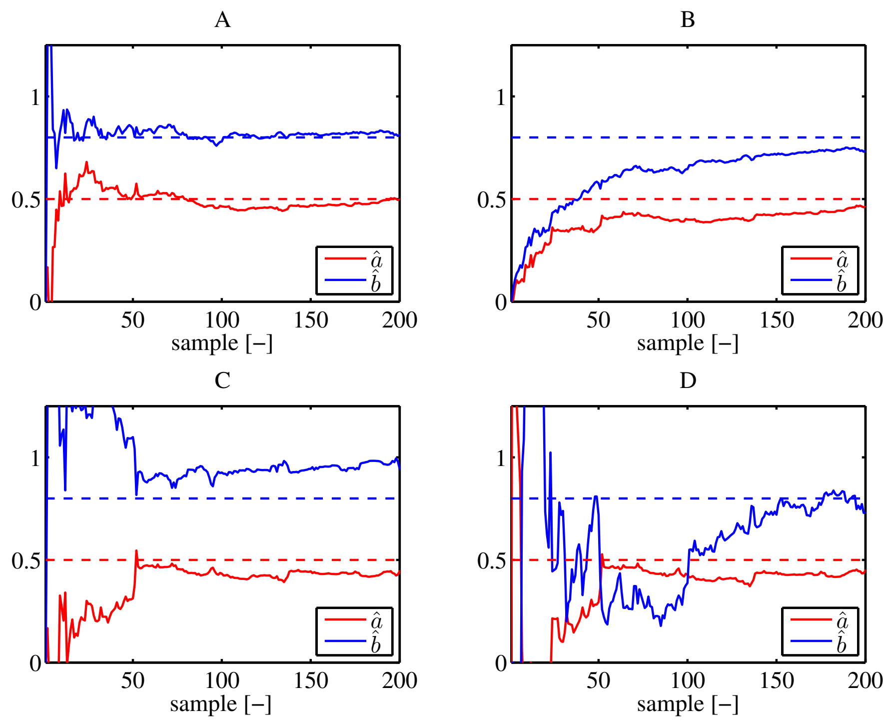

Figure 1 RLS Experiments

(4 p)

Solution

- D gives the most noisy estimates, D can thus be paired with 3 where the noice variance is high.

- C does not converge to the correct estimates, this can be explained by the insufficiently exciting input signal in 2 where u is constant.
- Now we have two cases left, with the same input and different P_0. A small P_0 gives a slower initial transient, A therefore belongs to 4 and B belongs to 1.

5. The system

$$G(q) = \frac{q+10}{q(q+0.6)}$$

is to be controlled using iterative learning control (ILC) according to the block diagram in Figure 2. The control signal is updated according to

$$u_{k+1}(t) = Q(q)(u_k(t) + L(q)e_k(t))$$

a. Explain the roles of the filters $Q(q)$ and $L(q)$ and why they do not have to be causal. (1.5 p)

b. Choose $Q = 1$ and $L = \kappa q(q + 0.6)$ and find a κ such that the ILC iterations are stable. (2 p)

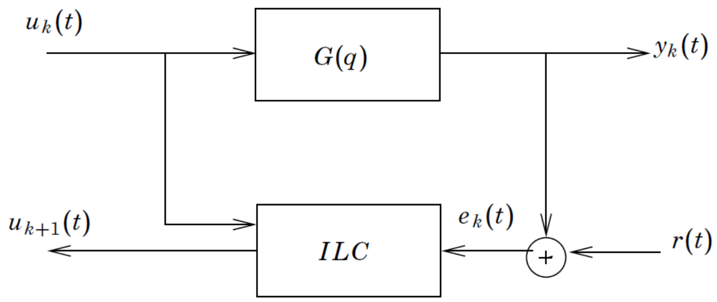

Figure 2 An ILC feedback system.

Solution

a. L is the corrective filter that updates u so that the error is counteracted whilst Q is a low-pass filter that is introduced for improved robustness and noise-insensitivity. The filters do not have to be causal since the filtering is done in-between (and not during) iterations where the whole error signal is available.

b. A sufficient condition for stability is given by

$$\sup_\omega |1 - L(i\omega)G(i\omega)| < 1$$

$$\begin{aligned}\sup_\omega |1 - L(i\omega)G(i\omega)| &= \sup_\omega |1 - \kappa\cos(wh) - \kappa i \sin(wh) - 10\kappa| \\ &= \sup_\omega \sqrt{(1-\kappa\cos(wh) - 10\kappa)^2 + \kappa^2\sin(wh)^2} \\ &= \sup_\omega \sqrt{(1-10\kappa)^2 - 2(1-10\kappa)\kappa\cos(wh) + \kappa^2}\end{aligned}$$

here we can see that $\kappa = 0.1$ gives

$$= \sqrt{(1-10\kappa)^2 - 2(1-10\kappa)\kappa\cos(wh) + \kappa^2} = \sqrt{\kappa^2} = 0.1 < 1$$

6.

a. Give an intuitive explanation for what a Lyapunov function is. Why can it be used to determine the stability of an arbitrary system? (1 p)

b. Consider the MRAS in Figure 3. Use passivity theory to show that the system interconnection is stable if $G(s)$ is the strictly positive real (SPR) system

$$G(s) = \frac{1}{s+1}$$

(3 p)

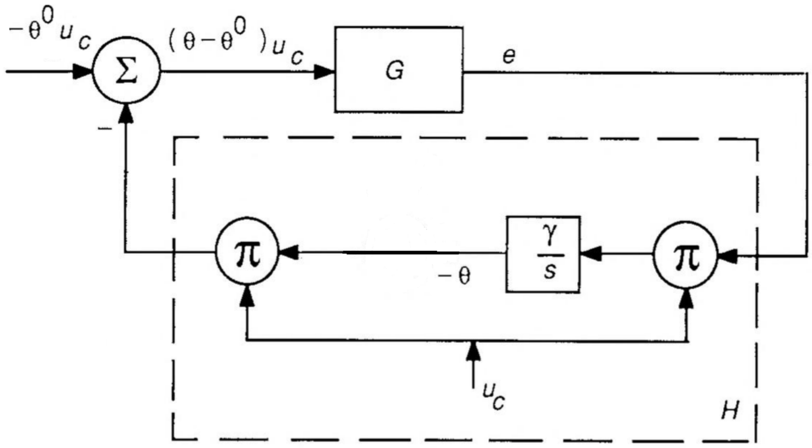

Figure 3 A model-reference adaptive system where the feedforward gain θ is adapted.

Solution

a. A Lyapunov function $V(x)$ describes a quantity in the system that is decreasing with time. We can think of it as an energy function, where the energy is a quantity present in the system that, if it decreases with time, ensures the stability of the system. If $V(x,u)$ contains a controlled variable (control input) u, we can construct a control law that modifies $V(x,u)$ such that it fulfills the requirements of a Lyapunov function. The formal requirements on a Lyapunov function are

- It increases radially with the state vector, i.e. when the state vector is large, the Lyapunov function is large.
- The time derivative $\dot{V}$ is non-positive, i.e. the function does not grow over time.
- It is zero at the origin.

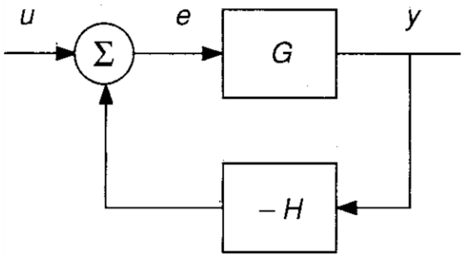

Figure 4 Feedback interconnection.

b. In Figure 3 we have a feedback interconnection between the systems G and H on the form displayed in Figure 4. If we can show that one of the systems is strictly passive and the other is passive, then the interconnection is stable. Since $G(s)$ is SPR and therefore strictly passive we only have to show that H with input e and output $-u_c\theta$ is passive.

We have that

$$\int_0^T yudt = \int_0^T -u_c\theta edt = \int_0^T u_c\frac{\gamma}{p}(u_ce)edt = [\omega = u_ce]$$
$$= \int_0^T w\frac{\gamma}{p}(w)dt \geq 0$$

where p is the derivative operator. This expression is ≥ 0 since $\frac{\gamma}{s}$ is positive real, H is therefore passive and the interconnection is stable.